# Chapter 3
# Group Velocity Dispersion

# Different Propagation Regions

$$i\frac{\partial A}{\partial z} = -\frac{i\alpha}{2}A + \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \gamma|A|^2 A,$$

- "*A*": **slowly varying amplitude** of the pulse envelope
- "*T*": measured in a **frame of reference** moving with the pulse at the **group velocity** $v_g$ ($T = t - z/v_g$).
- Effects of pulses propagating inside optical fibers.
  - Losses 、Dispersion 、Nonlinearity
- Depending on the **initial width** $T_0$ and the **peak power** $P_0$ of the incident pulse, either **dispersive** or **nonlinear effects** may dominate along the fiber.
- Dispersion length: $L_D$ 、Nonlinear length: $L_{NL}$
  - Depending on the relative magnitudes of $L_D$ 、$L_{NL}$ and the fiber length L, pulses can evolve quite differently.

# Different Propagation Regions

- Let us introduce a **time scale** normalized to the **input pulse** width $T_0$

$$\tau = \frac{T}{T_0} = \frac{t - z/v_g}{T_0}. \qquad (3.1.2)$$

- At the same time, we introduce a **normalized amplitude** $U$ .

$$A(z,\tau) = \sqrt{P_0}\exp(-\alpha z/2)U(z,\tau), \qquad (3.1.3)$$

$$i\frac{\partial A}{\partial z} = -\frac{i\alpha}{2}A + \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \gamma|A|^2 A, \qquad (3.1.1)$$

$$i\frac{\partial U}{\partial z} = \frac{\operatorname{sgn}(\beta_2)}{2L_D}\frac{\partial^2 U}{\partial \tau^2} - \frac{\exp(-\alpha z)}{L_{\rm NL}}|U|^2 U, \qquad (3.1.4)$$

- Where $\operatorname{sgn}(\beta_2) = \pm 1$ depending on the sign of the GVD parameter $\beta_2$

$$L_D = \frac{T_0^2}{|\beta_2|}, \qquad L_{\rm NL} = \frac{1}{\gamma P_0}. \qquad (3.1.5)$$

# Different Propagation Regions

- The **dispersion length** $L_D$ and the **nonlinear length** $L_{NL}$ provide the **length scales** over which **dispersive** or **nonlinear effects** become important for pulse evolution.
- Depending on the **relative magnitudes** of $L$, $L_D$, and $L_{NL}$, the propagation behavior can be classified in the following four categories.

1. $L \ll L_{NL}$ 、 $L \ll L_D$
2. $L \ll L_{NL}$ 、 $L \sim L_D$
3. $L \sim L_{NL}$ 、 $L \ll L_D$
4. $L \geq L_{NL}$ 、 $L \geq L_D$

# Different Propagation Regions

***L*** **<<** ***L*****NL** **、** ***L*** **<<** ***L*****D**

- The fiber plays a **passive role** in this regime and acts as a mere **transporter** of optical pulses.(except for reducing the **pulse energy** because of fiber **losses**).
- This regime is useful for **optical communication systems**.

Ex.

At **λ = 1550 nm**, $|\beta_2| \approx 20$ ps$^2$/km, and $\gamma \approx 2$ W$^{-1}$km$^{-1}$ for **standard telecommunication fibers** . ($T_0 > 100$ ps and $P_0 < 1$ mW)

$L_D = L_{NL} = 500$ km

$$L_D = \frac{T_0^2}{|\beta_2|}, \qquad L_{NL} = \frac{1}{\gamma P_0}. \qquad (3.1.5)$$

The **dispersive** and **nonlinear effects** are negligible for $L < 50$ km.

## $L \ll L_{NL}$ 、 $L \sim L_D$

- The **pulse evolution** is then **governed by GVD**, and the nonlinear effects play a relatively minor role.

$$\frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|} \ll 1. \qquad (3.1.6)$$

## $L \sim L_{NL}$ 、 $L \ll L_D$

- **Pulse evolution** in the fiber is **governed by SPM** that produces changes in the pulse spectrum.
- This phenomenon is considered in Chapter 4. **The nonlinearity-dominant regime** is applicable whenever

$$\frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|} \gg 1. \qquad (3.1.7)$$

## $L \gg L_{NL}$ 、 $L \gg L_D$

- **Dispersion** and **nonlinearity act together** as the pulse propagates along the fiber.

- The interplay of the **GVD** and **SPM effects** can lead to a qualitatively different behavior compared with that expected from **GVD** or SPM alone.

- This chapter is devoted to the linear regime, and the following discussion is applicable to pulses whose parameters satisfy

$$\frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|} \ll 1. \tag{3.1.6}$$

# 3.2 Dispersion-Induced Pulse Broadding

3.2.1 Gaussian Pulses

3.2.2 Chirped Gaussian Pulses

3.3.3 Hyperbolic Secant Pulses

3.3.4 Super-Gaussian Pulses

3.3.5 Experimental Results

- The effects of **GVD** on **optical pulses** propagating in a **linear dispersive medium** are studied by setting **γ=0** in Eq. (3.1.1).
- If we define the normalized amplitude *U*(z, *T*)according to Eq. (3.1.3), U(z, T) satisfies the following **linear partial differential equation**:

$$i\frac{\partial A}{\partial z} = -\frac{i\alpha}{2}A + \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \gamma|A|^2 A \qquad (3.1.1)$$

$$A(z,\tau) = \sqrt{P_0}\exp(-\alpha z/2)U(z,\tau), \qquad (3.1.3)$$

$$i\frac{\partial U}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 U}{\partial T^2} \qquad (3.2.1)$$

- This equation is similar to the **paraxial wave equation** that governs **diffraction** of **CW light** and becomes identical to it when **diffraction** occurs in only one **transverse direction**.
- But, $\boldsymbol{\beta_2}$ is replaced by $-\lambda/(2\pi)$.

- For above reason the **dispersion-induced temporal effects** have a close analogy with the **diffraction-induced spatial effects** .

$$i\frac{\partial U}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 U}{\partial T^2} \qquad (3.2.1)$$

- Equation (3.2.1) is **readily solved** by using the **Fourier-transform** method. If Ũ (z,ω)is the **Fourier transform** of U(z, *T*) such that

$$U(z,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(z,\omega)\exp(-i\omega T)\,d\omega \tag{3.2.2}$$

- It satisfies an ordinary differential equation.

$$i\frac{\partial \tilde{U}}{\partial z} = -\tfrac{1}{2}\beta_2\omega^2\tilde{U} \tag{3.2.3}$$

- The **solution** is given by：

$$\tilde{U}(z,\omega) = \tilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\omega^2 z\right) \tag{3.2.4}$$

- (3.2.4) shows that **GVD changes** the **phase** of each **spectral component** of the pulse by an amount that depends on both the **frequency** and the **propagated distance**.
- Even though such **phase changes do not** affect the **pulse spectrum**, they can modify the **pulse shape**.

$$\tilde{U}(z,\omega) = \tilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\omega^2 z\right) \qquad (3.2.4)$$

- ***Ũ*(0,ω)** is the **Fourier transform** of the incident field at **z=0** and is obtained using:

$$\tilde{U}(0,\omega) = \int_{-\infty}^{\infty} U(0,T)\exp(i\omega T)\,dT \qquad (3.2.5)$$

- Equations (3.2.5) and (3.2.6) can be used for **input pulses** of **arbitrary shapes**.

$$U(z,T)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\omega^2 z-i\omega T\right)d\omega \qquad (3.2.6)$$

$$U(z,T)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{U}(z,\omega)\exp(-i\omega T)\,d\omega \qquad (3.2.2)$$

$$\tilde{U}(z,\omega)=\tilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\omega^2 z\right) \qquad (3.2.4)$$

# Gaussian Pulses

- As a simple example, consider the case of a **Gaussian pulse** for which the incident field is of the form：

$$U(0,T) = \exp\left(-\frac{T^2}{2T_0^2}\right) \quad (3.2.7)$$

$T_0$: the half-width.

- The **full width at half maximum (FWHM)** in place of $\boldsymbol{T_0}$
- Their relation is：

$$T_{\text{FWHM}} = 2(\ln 2)^{1/2} T_0 \approx 1.665 T_0. \quad (3.2.8)$$

- We use Eqs. (3.2.5) through (3.2.7) and carry out the integration over ω using the **well-known identity** :

$$\int_{-\infty}^{\infty} \exp(-ax^2 + bx)\, dx = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{b^2}{4a}\right), \qquad (3.2.9)$$

$$U(0,T) = \exp\left(-\frac{T^2}{2T_0^2}\right), \qquad (3.2.7)$$

↓

$$\tilde{U}(0,\omega) = \int_{-\infty}^{\infty} U(0,T) \exp(i\omega T)\, dT. \qquad (3.2.6)$$

↓

$$U(z,T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{U}(0,\omega) \exp\left(\frac{i}{2}\beta_2 \omega^2 z - i\omega T\right) d\omega, \qquad (3.2.5)$$

- The amplitude $U(z, T)$ at any point $z$ along the fiber is given by：

$$U(z,T) = \frac{T_0}{(T_0^2 - i\beta_2 z)^{1/2}} \exp\left[-\frac{T^2}{2(T_0^2 - i\beta_2 z)}\right]. \quad (3.2.10)$$

- **Gaussian pulse** maintains its **shape** on propagation but its width $T_1$ increases with z as：

$$T_1(z) = T_0[1 + (z/L_D)^2]^{1/2}, \quad (3.2.11)$$

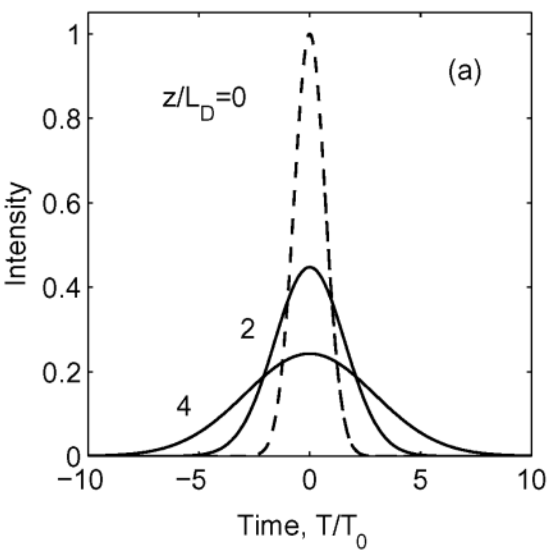
**Dispersion length:** $L_D = T_0^2 / |\beta_2|$

- Equation（3.2.11）shows how GVD **broadens** a Gaussian pulse.

For a given fiber length：

$$T_1(z) = T_0[1 + (z/L_D)^2]^{1/2}, \qquad (3.2.11)$$

- At **z =** $\boldsymbol{L_D}$, a **Gaussian pulse** broadens by a factor of $\sqrt{2}$
- For a given fiber length, **short pulses** broaden more because of a **smaller dispersion length**. (**Dispersion length:** $\boldsymbol{L_D = T_0^{\ 2}/|\beta_2|}$)
- It shows the extent of dispersion-induced broadening for a Gaussian pulse in the fiber.




Normalized intensity $|U|^2$ and as functions of $T/T_0$ for a Gaussian pulse at z = $2L_D$ and $4L_D$.

- A comparison of Eqs. (3.2.7) and (3.2.10) shows that although the incident pulse is unchirped (with no phase modulation), the transmitted pulse becomes chirped.

$$U(0,T) = \exp\left(-\frac{T^2}{2T_0^2}\right), \qquad (3.2.7)$$

$$U(z,T) = \frac{T_0}{(T_0^2 - i\beta_2 z)^{1/2}} \exp\left[-\frac{T^2}{2(T_0^2 - i\beta_2 z)}\right]. \qquad (3.2.10)$$

- It can be seen clearly by writing $U(z, T)$ in the form：

$$U(z,T) = |U(z,T)| \exp[i\phi(z,T)] \qquad (3.2.12)$$

where

$$\phi(z,T) = -\frac{\mathrm{sgn}(\beta_2)(z/L_D)}{1+(z/L_D)^2}\frac{T^2}{2T_0^2} + \frac{1}{2}\tan^{-1}\left(\frac{z}{L_D}\right). \qquad (3.2.13)$$

- The **phase** depends on whether the pulse experiences **normal** or **anomalous dispersion** inside the fiber.

$$\phi(z,T) = -\frac{\mathrm{sgn}(\beta_2)(z/L_D)}{1+(z/L_D)^2}\frac{T^2}{2T_0^2} + \frac{1}{2}\tan^{-1}\left(\frac{z}{L_D}\right). \qquad (3.2.13)$$

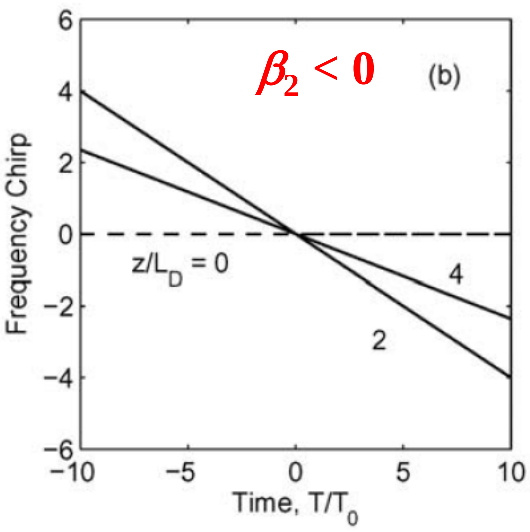
- The **time dependence** of $\phi(z,T)$ implies that the **instantaneous frequency differs** across the pulse from the **central frequency** $\omega_0$.
- The difference $\delta\omega$ is just the derivative $-\partial\phi/\partial T$

$$\delta\omega(T) = -\frac{\partial\phi}{\partial T} = \frac{\mathrm{sgn}(\beta_2)(z/L_D)}{1+(z/L_D)^2}\frac{T}{T_0^2}. \qquad (3.2.14)$$

- The **different frequency components** of a pulse travel at **slightly different speeds** along the fiber because of GVD.
- The **chirp δω** depends on the **sign** of $\beta_2$.
- In the **normal-dispersion regime** ($\beta_2 > 0$), δω is **negative** at the leading edge (***T*** < 0) and **increases linearly** across the pulse.

$$\delta\omega(T) = -\frac{\partial\phi}{\partial T} = \frac{\mathrm{sgn}(\beta_2)(z/L_D)}{1+(z/L_D)^2}\frac{T}{T_0^2}. \tag{3.2.14}$$

- In the **anomalous dispersion regime** ($\beta_2 < 0$), *δω* **decreases linearly** across the pulse.(below figure)




Frequency chirp δω as functions of $T/T_0$ for a Gaussian pulse at z = $2L_D$ and $4L_D$

- More specifically, **red components travel faster** than blue components in the **normal-dispersion regime** ($\beta_2$>0).
- While the **opposite** occursin the **anomalous-dispersion regime** ($\beta_2$<0).
- The pulse can maintain its width only **if all spectral components arrive together**.
- Any **time delay** in the arrival of **different spectral components** leads to **pulse broadening**.

# Chirped Gaussian Pulses

- For an **initially unchirped Gaussian pulse**, Eq. (3.2.11) shows that **dispersion-induced broadening** of the pulse **does not** depend on the sign of the GVD parameter $\beta_2$.

$$T_1(z) = T_0[1 + (z/L_D)^2]^{1/2}, \qquad (3.2.11)$$

- For a given value of the **dispersion length** $L_D$, the pulse broadens by the **same amount** in the **normal-** and **anomalous-**dispersion regimes of the fiber.
- This behavior changes if the Gaussian pulse has an **initial frequency chirp**
- In the case of **linearly chirped Gaussian pulses**, the incident field can be written as：

$$U(0,T) = \exp\left[-\frac{(1+iC)}{2}\frac{T^2}{T_0^2}\right], \qquad (3.2.15)$$

$C$ : a chirp parameter

- Through Eq. (3.2.12) one finds that the **instantaneous frequency**
  - **increases linearly** from the leading to the trailing edge (**up-chirp**) for ***C*** **> 0**,
  - while the opposite occurs (**down-chirp**) for ***C*** **< 0**.

$$U(z,T) = |U(z,T)| \exp[i\phi(z,T)], \tag{3.2.12}$$

- It is common to **refer to** the **chirp** as being **positive** or **negative**, depending on whether ***C*** is positive or negative

- The numerical value of ***C*** can be estimated from the **spectral width** of the Gaussian pulse.

- By substituting Eq. (3.2.15) in Eq. (3.2.6) and using Eq. (3.2.9), $\tilde{U}(0, \omega)$ is given by：

$$U(0,T) = \exp\left[-\frac{(1+iC)}{2}\frac{T^2}{T_0^2}\right], \quad (3.2.15)$$

$$\tilde{U}(0,\omega) = \int_{-\infty}^{\infty} U(0,T)\exp(i\omega T)\,dT. \quad (3.2.6)$$

$$\tilde{U}(0,\omega) = \left(\frac{2\pi T_0^2}{1+iC}\right)^{1/2} \exp\left[-\frac{\omega^2 T_0^2}{2(1+iC)}\right]. \quad (3.2.16)$$

- The **spectral half-width** from eq.(3.2.16) is given by ：

$$\Delta\omega = (1+C^2)^{1/2}/T_0. \quad (3.2.17)$$

- In the absence of **frequency chirp** ($C$ = 0), the spectral width is **transform-limited** and satisfies the relation $\Delta\omega T_0 = 1$.

- Through eq. (3.2.17) we can know that the **spectral width** of a pulse is enhanced by a factor of $\mathbf{(1+C^2)^{1/2}}$ in the presence of **linear chirp**.

$$\Delta\omega = (1 + C^2)^{1/2}/T_0. \qquad (3.2.17)$$

- Equation (3.2.17) can be used to estimate | $C$ | from measurements of $\Delta\omega$ and $T_0$.
- To obtain the **transmitted field**, $\tilde{U}(0,\omega)$ from Eq. (3.2.16) is **substituted in** Eq. (3.2.5).

$$\tilde{U}(0,\omega) = \left(\frac{2\pi T_0^2}{1+iC}\right)^{1/2} \exp\left[-\frac{\omega^2 T_0^2}{2(1+iC)}\right]. \qquad (3.2.16)$$

$$U(z,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\omega^2 z - i\omega T\right)d\omega, \qquad (3.2.5)$$

- The **integration** can again be **performed analytically** using Eq. (3.2.9) with the result：

$$U(z,T)=\frac{T_0}{[T_0^2-i\beta_2 z(1+iC)]^{1/2}}\exp\left(-\frac{(1+iC)T^2}{2[T_0^2-i\beta_2 z(1+iC)]}\right). \quad (3.2.18)$$

- Even a **chirped Gaussian pulse** maintains its **Gaussian shape** on propagation. The width $T_1$ after propagating a distance z is related to the **initial width** $T_0$ by the relation ：

$$\frac{T_1}{T_0}=\left[\left(1+\frac{C\beta_2 z}{T_0^2}\right)^2+\left(\frac{\beta_2 z}{T_0^2}\right)^2\right]^{1/2}. \quad (3.2.19)$$

- The **chirp parameter** of the pulse also changes from $C$ to $C_1$ such that

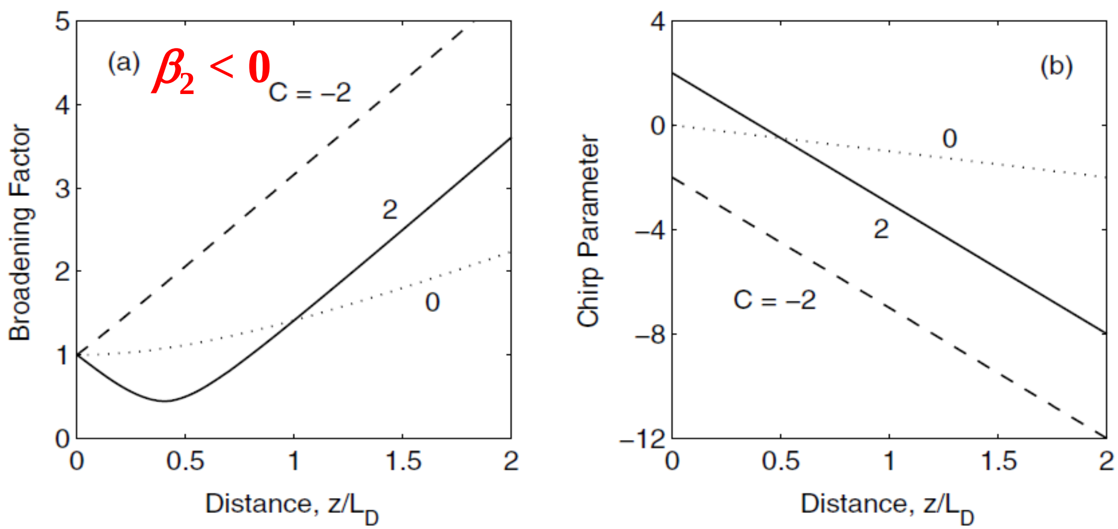
$$C_1(z)=C+(1+C^2)(\beta_2 z/T_0^2). \quad (3.2.20)$$

- It is useful to define a **normalized distance** $\xi$ as $\xi = z/L_D$,

$$L_D = \frac{T_0^2}{|\beta_2|},$$

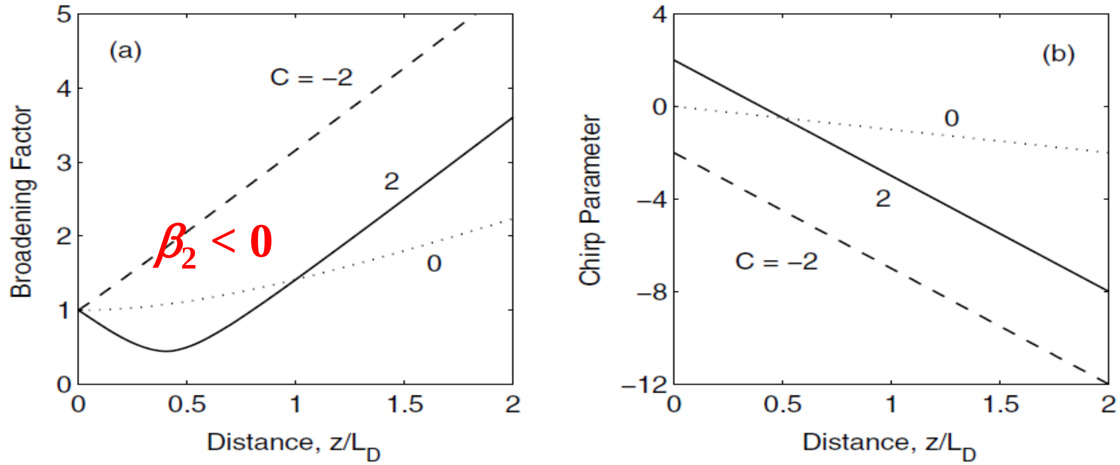
- Figure 3.2 shows (a) the **broadening factor** $T_1/T_0$ and (b) the chirp parameter $C_1$ as a function of $\xi$ in the case of **anomalous dispersion** ($\beta_2 < 0$).
- An **unchirped pulse** ($C = 0$) **broadens monotonically** by a factor of $(1+\xi^2)^{1/2}$ and develops a **negative chirp** such that $C_1 = -\xi$ (the dotted curves).

$$\frac{T_1}{T_0} = \left[\left(1 + \frac{C\beta_2 z}{T_0^2}\right)^2 + \left(\frac{\beta_2 z}{T_0^2}\right)^2\right]^{1/2}. \qquad (3.2.19)$$



- **Chirped pulses**, on the other hand, may **broaden** or **compress depending on whether** $\beta_2$ and $C$ have the **same** or **opposite signs**.
- When $\boldsymbol{\beta_2 C > 0}$, a **chirped Gaussian pulse** broadens **monotonically** at a rate faster than that of the **unchirped pulse** (the dashed curves).
- The reason is **related to the fact** that the **dispersion-induced chirp** adds to the **input chirp** because the two contributions have the same sign.

$$C_1(z) = C + (1 + C^2)(\beta_2 z / T_0^2). \qquad (3.2.20)$$




Broadening factor (a) and the chirp parameter (b) as functions of distance for a chirped Gaussian pulse propagating in the anomalous-dispersion region of a fiber.

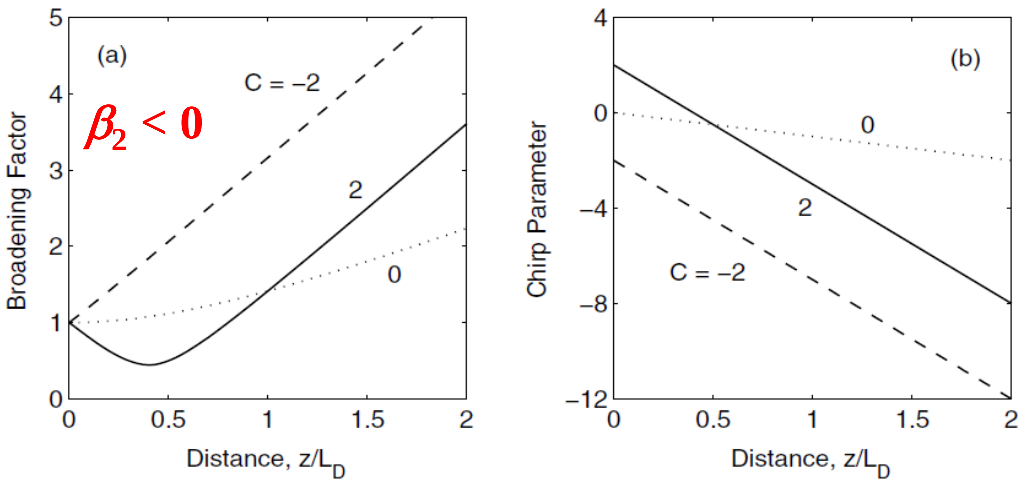
- The situation changes dramatically for $\boldsymbol{\beta_2 C < 0}$. $C_1$ becomes zero at a distance $\xi = |C|/(1+C^2)$, and the pulse becomes unchirped.

$$\frac{T_1}{T_0} = \left[\left(1 + \frac{C\beta_2 z}{T_0^2}\right)^2 + \left(\frac{\beta_2 z}{T_0^2}\right)^2\right]^{1/2}. \qquad (3.2.19)$$

- The **minimum value** of the **pulse width** depends on the input chirp parameter as

$$T_1^{\min} = \frac{T_0}{(1+C^2)^{1/2}}. \qquad (3.2.21)$$

- Since $C_1 = 0$ when the pulse attains its minimum width, it becomes **transform-limited** such that $\Delta\omega_0 T^{\min}{}_1 = 1$, where $\Delta\omega_0$ is the input spectral width of the pulse.



# Hyperbolic Secant Pulses

- Although pulses emitted from many lasers can be approximated by a **Gaussian shape**, it is necessary to consider other pulse shapes.

- The **hyperbolic secant pulse shape** occurs naturally in the context of optical solitons and pulses emitted from some mode-locked lasers.

- The **optical field** associated with such pulses often takes the form :

$$U(0,T) = \operatorname{sech}\left(\frac{T}{T_0}\right)\exp\left(-\frac{iCT^2}{2T_0^2}\right) \qquad (3.2.22)$$

C (Chirp parameter): controls the initial chirp .

- The **transmitted field** *U* (z , *T*) is obtained by using Eq. (3.2.5), (3.2.6), and (3.2.22).

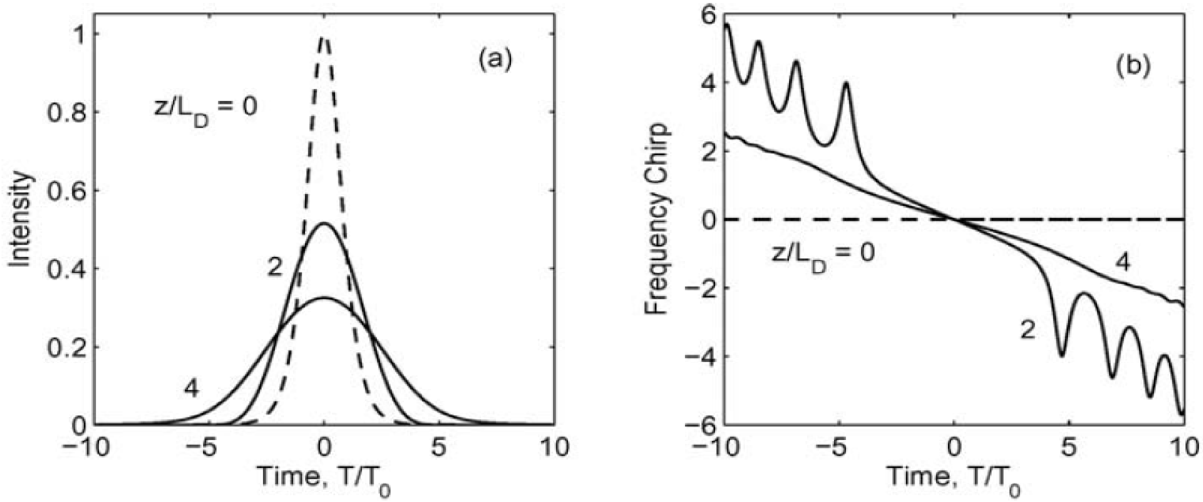
$$U(0,T) = \mathrm{sech}\left(\frac{T}{T_0}\right)\exp\left(-\frac{iCT^2}{2T_0^2}\right) \qquad (3.2.5)$$

$$\tilde{U}(0,\omega) = \int_{-\infty}^{\infty} U(0,T)\exp(i\omega T)\,dT \qquad (3.2.6)$$

$$U(z,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\omega^2 z - i\omega T\right)d\omega \qquad (3.2.22)$$

- Figure 3.3 shows the intensity and **chirp profiles** calculated numerically at $\boldsymbol{z = 2L_D}$ and $\boldsymbol{z = 4L_D}$ for initially **unchirped pulses** ($C$ = 0).
- A comparison of Figures 3.1 and 3.3 shows that the **qualitative features** of **dispersion-induced broadening** are nearly **identical** for the **Gaussian** and "**sech**" pulses.
- The **main difference** is that the **dispersion-induced chirp** is no longer **purely linear** across the pulse.

**Figure 3.3**



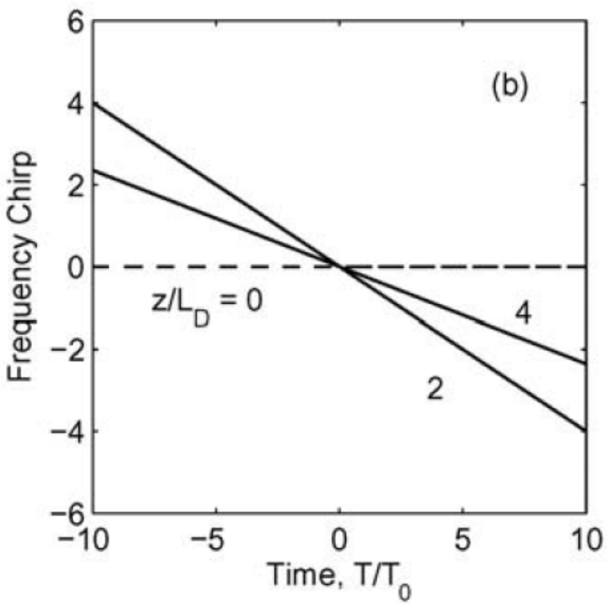

Normalized (a) intensity $|U|^2$ and (b) frequency chirp $\delta\omega T_0$ as a function of T/T0 for a "sech" pulse at z = $2L_D$ and $4L_D$. Dashed lines show the input profiles at z = 0.

- Note that $T_0$ appearing in Eq. (3.2.22) is not the FWHM but is related to it by：

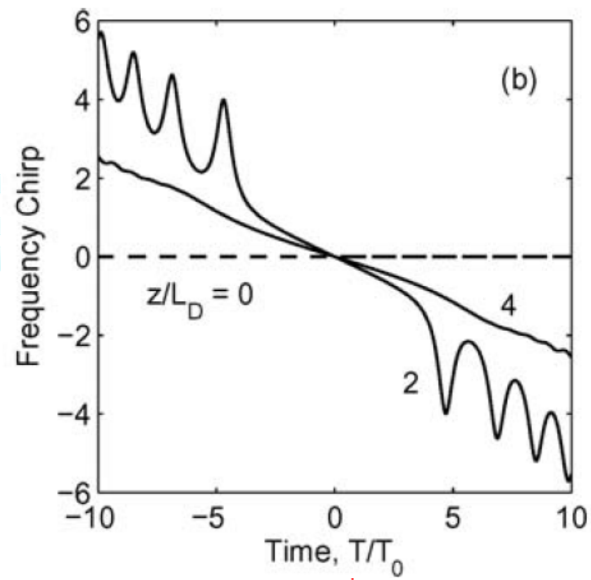
$$U(0,T) = \operatorname{sech}\left(\frac{T}{T_0}\right)\exp\left(-\frac{iCT^2}{2T_0^2}\right) \qquad (3.2.22)$$

$$T_{\mathrm{FWHM}} = 2\ln(1+\sqrt{2})T_0 \approx 1.763\,T_0 \qquad (3.2.23)$$

**Chirp of Gaussian pulse**




**Chirp of Sech pulse**



# Super-Gaussian Pulses

- As one may expect, **dispersion-induced broadening** is sensitive to the **steepness** of pulse edges.
- In general, a pulse with **steeper leading** and **trailing edges broadens more rapidly** with propagation simply because such a pulse has a **wider spectrum** to start with.
- A **super-Gaussian shape** can be used to **model the effects** of **steep leading** and **trailing edges** on dispersion-induced pulse broadening.

- For a **super-Gaussian pulse**, Eq. (3.2.15) is generalized to take the form ：

$$U(0,T) = \exp\left[-\frac{1+iC}{2}\left(\frac{T}{T_0}\right)^{2m}\right] \qquad (3.2.24)$$

- The parameter m controls the degree of edge sharpness.
  - For ***m*** **= 1** we recover the case of **chirped Gaussian pulses**.
  - For **larger value of m**, the pulse becomes **square shaped** with **sharper leading** and trailing edges.

- If the **rise time** $T_r$ is defined as the **duration** during which the intensity increases from **10 to 90%** of its peak value, it is related to the parameter m as ：

$$T_r = (\ln 9)\frac{T_0}{2m} \approx \frac{T_0}{m}. \qquad (3.2.25)$$

- Thus the parameter m can be determined from the measurements of $T_r$ and $T_0$.

- Figure 3.4 shows the intensity and chirp profiles at **z = 2$L_D$**, and 4$L_D$ in the case of an initially unchirped **super-Gaussian pulse** (C = 0) by using ***m* = 3**.
- It should be compared with Figure 3.1 where the case of a Gaussian pulse (*m* = 1) is shown.
- The differences between the two can be attributed to the **steeper leading** and **trailing edges** associated with a **Super-Gaussian pulse**.

**Gaussian** **super-Gaussian pulse**

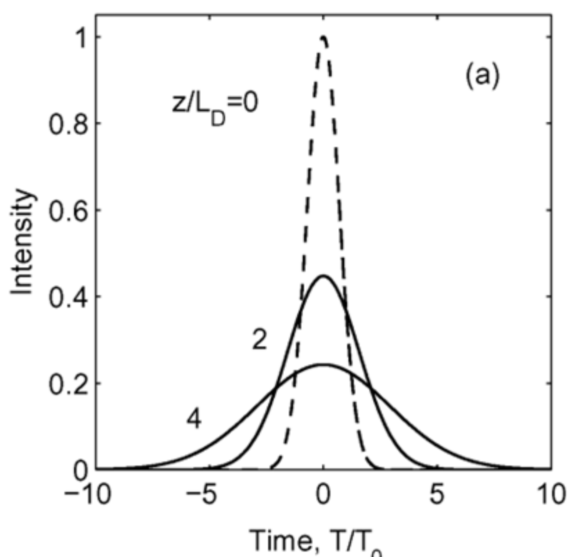



Figure 3.1

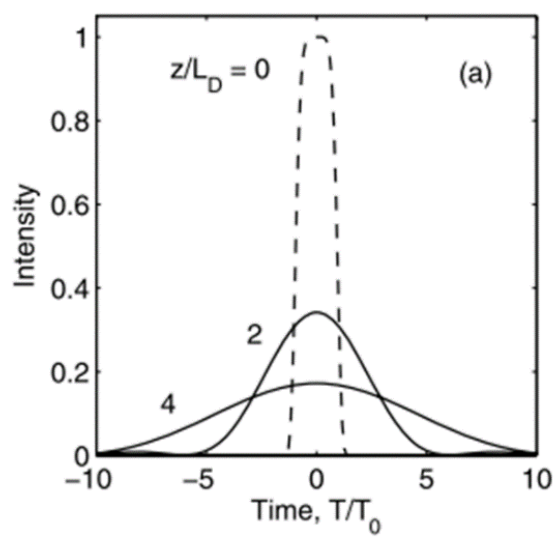
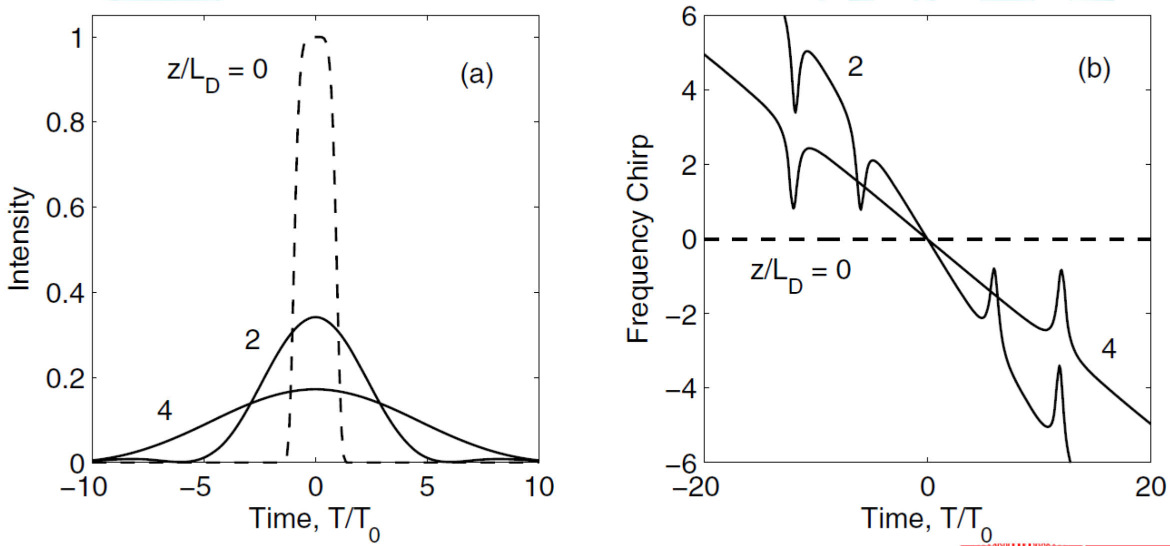



Figure 3.4

- Whereas the **Gaussian pulse** maintains its shape during propagation, the super-Gaussian pulse **not only broadens** at a faster rate but is also **distorted in shape**.
- The **chirp profile** is also far from being **linear** and exhibits **high-frequency oscillations**
- Enhanced broadening of a **super-Gaussian pulse** can be understood by noting that its **spectrum** is wider than that of a **Gaussian pulse** because of **steeper leading** and **trailing edges**.
- As the **GVD-induced delay** of each frequency component is directly related to its separation from the **central frequency** $\omega_0$, a **wider spectrum** results in a faster rate of pulse broadening.



- For **complicated pulse shapes** such as those seen in Figure 3.4, the FWHM is **not** a **true measure** of the pulse width.
- The **width of such pulses** is more accurately described by the **root-mean-square (RMS)** width $\sigma$ defined as

$$\sigma = [\langle T^2 \rangle - \langle T \rangle^2]^{1/2}, \qquad (3.2.26)$$

The **angle brackets** denote **averaging** over the intensity profile as

$$\langle T^n \rangle = \frac{\int_{-\infty}^{\infty} T^n |U(z,T)|^2 \, dT}{\int_{-\infty}^{\infty} |U(z,T)|^2 \, dT}. \qquad (3.2.27)$$

- The moments $\langle T \rangle$ and $\langle T^2 \rangle$ can be calculated analytically for some specific cases.

- It is possible to evaluate the **broadening factor $\sigma/\sigma_0$** analytically for **super-Gaussian pulses** using Eqs. (3.2.5) and (3.2.24) through (3.2.27) with the result

$$\frac{\sigma}{\sigma_0} = \left[1 + \frac{\Gamma(1/2m)}{\Gamma(3/2m)}\frac{C\beta_2 z}{T_0^2} + m^2(1+C^2)\frac{\Gamma(2-1/2m)}{\Gamma(3/2m)}\left(\frac{\beta_2 z}{T_0^2}\right)^2\right]^{1/2}, \qquad (3.2.28)$$

  - where $\Gamma(x)$ is the gamma function.
- For a Gaussian pulse ($m$ = 1) the broadening factor reduces to that given in Eq. (3.2.19).

$$\frac{T_1}{T_0} = \left[\left(1 + \frac{C\beta_2 z}{T_0^2}\right)^2 + \left(\frac{\beta_2 z}{T_0^2}\right)^2\right]^{1/2}. \qquad (3.2.19)$$

- Figure 3.5 shows the broadening factor $\sigma/\sigma_0$ of **super-Gaussian pulses** as a function of the propagation distance for values of m ranging **from 1 to 4**.
- The case **m = 1** corresponds to a **Gaussian pulse**; the **pulse edges** become **increasingly steeper** for **larger values of m**.
- Noting from Eq. (3.2.25) that the **rise time** is inversely proportional to m, it is evident that a pulse with a **shorter rise time** broadens faster.
- The curves in Figure 3.5 are drawn for the case of **initially chirped pulses** with C = 5.

$$T_r = (\ln 9)\frac{T_0}{2m} \approx \frac{T_0}{m}. \qquad (3.2.25)$$

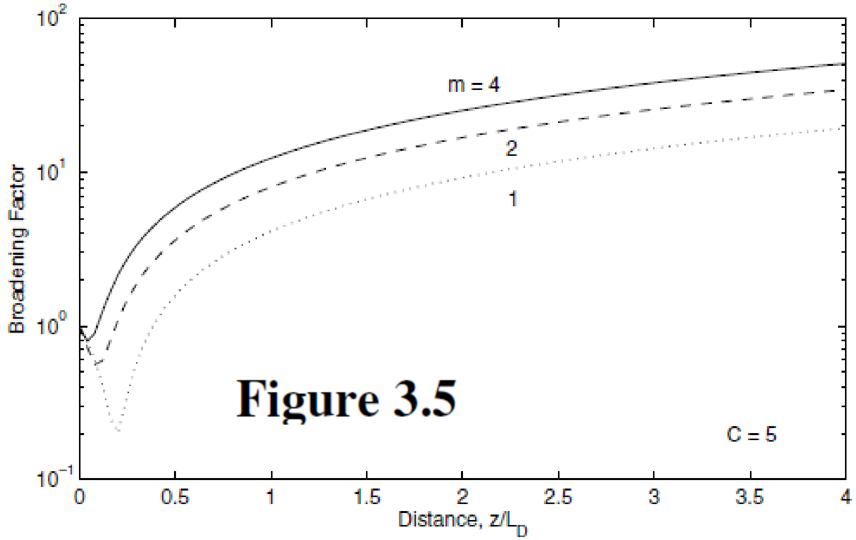



**Figure 3.5**

# 3.3 Third-Order Dispersion

- 3.3.1 Evolution of Chirped Gaussian Pulses
- 3.3.2 Broadening Factor

# 3.3 Third-Order Dispersion

- The **dispersion-induced pulse broadening** discussed in Section 3.2 is due to the lowest order **GVD** term proportional to $\boldsymbol{\beta_2}$ in Eq. (2.3.23).

$$\beta(\omega) = \beta_0 + (\omega - \omega_0)\beta_1 + \frac{1}{2}(\omega - \omega_0)^2\beta_2 + \frac{1}{6}(\omega - \omega_0)^3\beta_3 + \cdots, \qquad (2.3.23)$$

- The **third-order dispersion** (TOD) governed by $\beta_3$.

- If the **pulse wavelength** nearly coincides with the **zero-dispersion wavelength** $\lambda_D$ and $\boldsymbol{\beta_2 \approx 0}$, the $\boldsymbol{\beta_3}$ **term** provides the dominant contribution to the GVD effects [6].

- For ultrashort pulses (with width $T_0 < 1$ ps), **it is necessary to** include the $\boldsymbol{\beta_3}$ **term** even when $\boldsymbol{\beta_2 \neq 0}$ because the expansion parameter $\Delta\omega/\omega_0$ **is no longer** small enough to justify the truncation of the expansion in Eq. (2.3.23) after the $\beta_2$ term.

- This section considers the **dispersive effects** by including both $\beta_2$ and $\beta_3$ terms while still **neglecting the nonlinear effects**.
- The appropriate propagation equation for the amplitude $A(z,T)$ is obtained from *Eq.* (2.3.43) after setting $\gamma = 0$.

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2}A + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} = i\gamma\left(|A|^2 A + \frac{i}{\omega_0}\frac{\partial}{\partial T}(|A|^2 A) - T_R A\frac{\partial |A|^2}{\partial T}\right) \qquad (2.3.43)$$

- Using Eq. (3.1.3), $U(z, T)$ satisfies the following equation:

$$A(z,\tau) = \sqrt{P_0}\exp(-\alpha z/2)U(z,\tau), \qquad (3.1.3)$$

$U(z,\tau)$: **normalized amplitude**

$$i\frac{\partial U}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 U}{\partial T^2} + \frac{i\beta_3}{6}\frac{\partial^3 U}{\partial T^3}. \qquad (3.3.1)$$

- This equation can also be solved by using the **Fourier-transform** technique of Section3.2.
- In place of Eq. (3.2.5) the **transmitted field** is obtained from

$$U(z,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(z,\omega)\exp(-i\omega T)\,d\omega$$

$$U(z,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\omega^2 z - i\omega T\right)d\omega, \qquad (3.2.5)$$

$$U(z,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\omega^2 z + \frac{i}{6}\beta_3\omega^3 z - i\omega T\right)d\omega, \qquad (3.3.2)$$

- The **Fourier transform** *U (0,ω)* of the **incident field** is given by Eq. (3.2.6).

$$\tilde{U}(0,\omega) = \int_{-\infty}^{\infty} U(0,T)\exp(i\omega T)\,dT. \qquad (3.2.6)$$

- Equation (3.3.2) can be used to study the effect of **higher-order dispersion** if the **incident field** $U(0,T)$ is specified.
- In particular, one can consider **Gaussian**, **super-Gaussian**, or **hyperbolic-secant pulses** in a manner **analogous to** Section 3.2.
- As an analytic solution in terms of the **Airy functions** can be obtained for Gaussian pulses [6], we consider this case first.

# 3.3.1 Evolution of Chirped Gaussian Pulses

- In the case of a **chirped Gaussian pulse**, we use $U(0,\omega)$ from *Eq.* (3.2.16) in *Eq.* (3.3.2) and introduce $\mathbf{x = \omega p}$ as a **new integration variable**, where

$$\tilde{U}(0,\omega) = \left(\frac{2\pi T_0^2}{1+iC}\right)^{1/2} \exp\left[-\frac{\omega^2 T_0^2}{2(1+iC)}\right]. \qquad (3.2.16)$$

$$U(z,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\omega^2 z + \frac{i}{6}\beta_3\omega^3 z - i\omega T\right) d\omega, \qquad (3.3.2)$$

$$p^2 = \frac{T_0^2}{2}\left(\frac{1}{1+iC} - \frac{i\beta_2 z}{T_0^2}\right). \qquad (3.3.3)$$

- We then obtain the following expression:

$$U(z,T) = \frac{A_0}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\left(-x^2 + \frac{ib}{3}x^3 - \frac{iT}{p}x\right) dx, \qquad (3.3.4)$$

where $b = \beta^3 z/(2p^3)$.

- The $x^2$ term can be eliminated with another transformation $\mathbf{x = b^{-1/3}u - i/b}$.

- The **resulting integral** can be written in terms of the **Airy function** Ai(x) as

$$U(z,T) = \frac{2A_0\sqrt{\pi}}{|b|^{1/3}} \exp\left(\frac{2p-3bT}{3pb^2}\right) \mathrm{Ai}\left(\frac{p-bT}{p|b|^{4/3}}\right). \quad (3.3.5)$$

***p*** depends on the **fiber** and **pulse parameters** as indicated in Eq. (3.3.3).

$$p^2 = \frac{T_0^2}{2}\left(\frac{1}{1+iC} - \frac{i\beta_2 z}{T_0^2}\right). \quad (3.3.3)$$

- For an **unchirped pulse** whose spectrum is centered exactly at the **zero-dispersion wavelength** of the fiber (**$\beta_2$ = 0), p = $T_0/\sqrt{2}$**.
- As one may expect, pulse evolution along the fiber depends on the **relative magnitudes** of $\beta_2$ and $\beta_3$.
- To compare the **relative importance of** the $\beta_2$ and $\beta_3$ terms in Eq. (3.3.1), it is useful to introduce a **dispersion length** associated with the **TOD** as

$$L'_D = T_0^3/|\beta_3|. \quad (3.3.6)$$

$$L_D = \frac{T_0^2}{|\beta_2|}$$

- The **TOD effects** play a significant role only if $L'_D \leq L_D$ or $T_0|\beta_2/\beta_3| \leq 1$.
- For a 100-ps pulse, this condition implies that $\beta_2 < 10^{-3}$ ps$^2$/km when $\beta_3 = 0.1$ ps$^3$/km.
- Such low values of $\beta_2$ are realized only if $\lambda_0$ and $\lambda_D$ differ by <0.01 nm.
- In practice, it is difficult to match $\lambda_0$ and $\lambda_D$ to such an accuracy, and the contribution of $\beta_3$ is generally negligible compared with that of $\beta_2$.

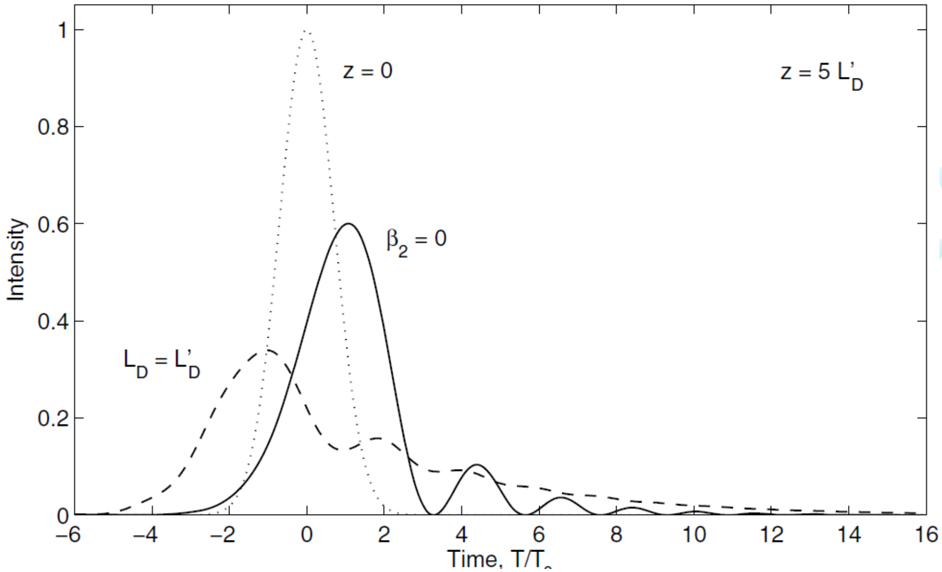
$$L'_D = T_0^3/|\beta_3|.$$

$$L_D = \frac{T_0^2}{|\beta_2|}$$

→

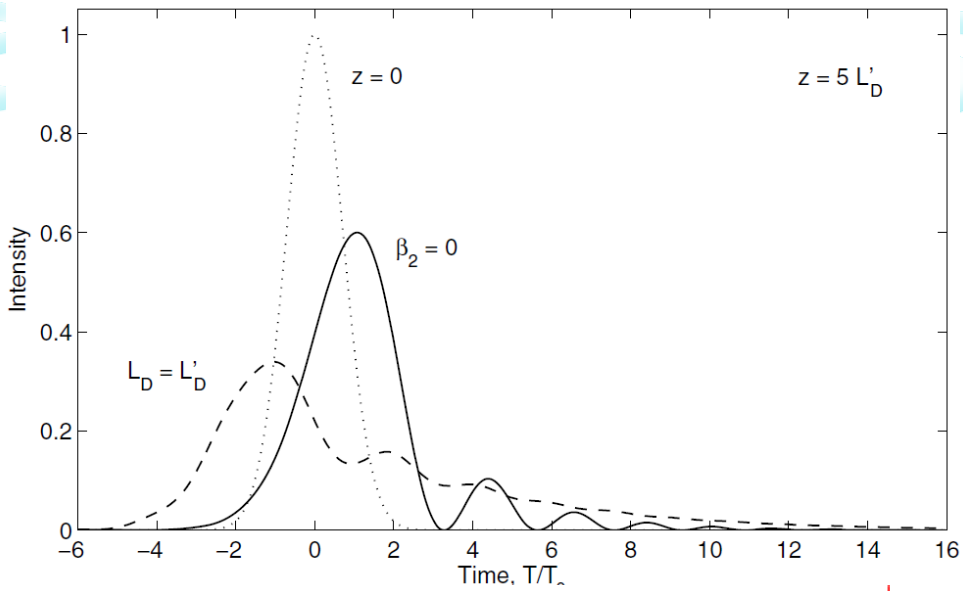
$$\boldsymbol{L`_D/L_D = T_0|\beta_2/\beta_3| \leq 1}$$

- Figure 3.6 shows the pulse shapes at z = $5L'_D$ for an **initially unchirped Gaussian pulse** (C = 0) for $\boldsymbol{\beta_2}$ **= 0 (solid curve**) and for a value of $\beta_2$ such that $L_D = L'_D$ (dashed curve).
- Whereas a **Gaussian pulse** remains Gaussian when only the $\boldsymbol{\beta_2}$ term in Eq. (3.3.1) contributes to **GVD** (Figure 3.1), the **TOD** distorts the pulse such that it becomes **asymmetric** with an **oscillatory structure** near one of its edges.
- When $\boldsymbol{\beta_3}$ is negative, it is the **leading edge** of the pulse that develops **oscillations**




- Pulse shapes at **z = 5$L'_D$** of an initially **Gaussian pulse** at z = 0 (**dotted curve**) in the presence of higher-order dispersion.
- **Solid curve** is for the case of $\boldsymbol{\lambda_0 = \lambda_D}$.
- **Dashed curve** shows the effect of finite $\beta_2$ in the case of $\boldsymbol{L_D = L'_D}$ .

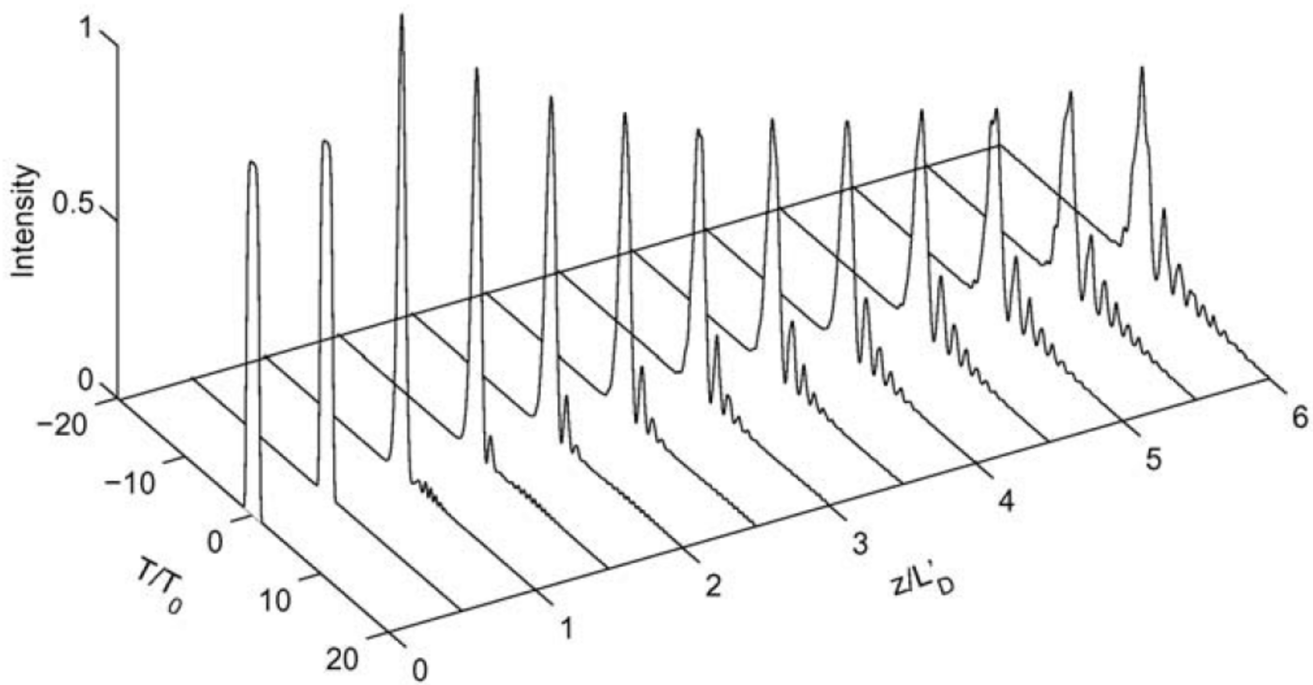
- When $\boldsymbol{\beta_2 = 0}$, oscillations are **deep**, with intensity **dropping to zero** between successive oscillations.
- However, these oscillations **damp significantly** even for relatively small values of $\beta_2$.
- For the case $\boldsymbol{L_D = L'_D}$ shown in Figure 3.6 ($\beta_2 = \beta_3/T_0$), oscillations have nearly disappeared, and the pulse has a **long tail** on the trailing side.
- For larger values of $\beta_2$ such that $\boldsymbol{L_D \ll L'_D}$, the pulse shape becomes **nearly Gaussian** as the TOD plays a relatively minor role.



- Equation (3.3.2) can be used to study **pulse evolution** for other pulse shapes although the Fourier transform must be performed numerically.

$$U(z,T)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{U}(0,\omega)\exp\left(\frac{i}{2}\beta_2\omega^2 z+\frac{i}{6}\beta_3\omega^3 z-i\omega T\right)d\omega, \qquad (3.3.2)$$

- Figure 3.7 shows evolution of an **unchirped super-Gaussian** pulse at the **zero-dispersion wavelength** ($\beta_2$ = 0) with **C = 0** and ***m* = 3** in Eq. (3.2.24).



- It is clear that pulse shapes can **vary widely** depending on the **initial conditions**.
- In practice, one is often interested in the extent of **dispersion-induced broadening** rather than details of pulse shapes.
- As the **FWHM** is not a true measure of the width of pulses shown in Figures 3.6 and 3.7, we use the RMS width σ defined in Eq. (3.2.26).

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- The width of such pulses is more accurately described by the root-mean-square (RMS) width σ defined as

$$\sigma = [\langle T^2 \rangle - \langle T \rangle^2]^{1/2}, \qquad (3.2.26)$$

where the **angle brackets denote** averaging over the intensity profile as

$$\langle T^n \rangle = \frac{\int_{-\infty}^{\infty} T^n |U(z,T)|^2 \, dT}{\int_{-\infty}^{\infty} |U(z,T)|^2 \, dT}. \qquad (3.2.27)$$

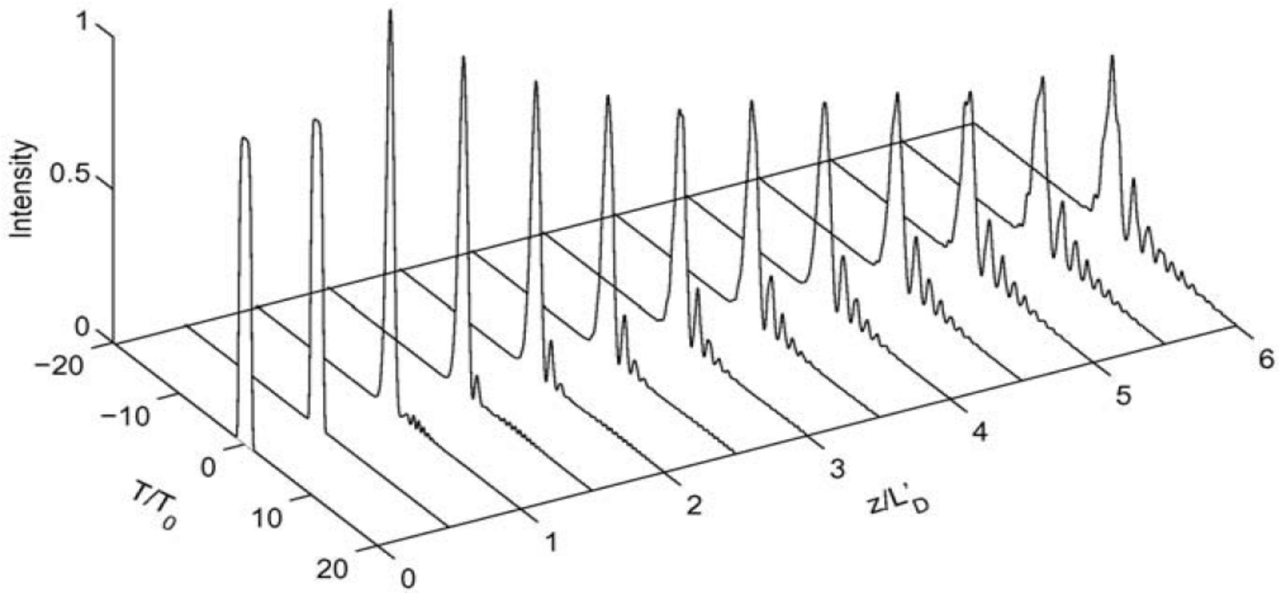


**Figure 3.7**: Evolution of a super-Gaussian pulse with $m = 3$ along the fiber length for the case of $\beta_2 = 0$ and $\beta_3 > 0$. Third-order dispersion is responsible for the oscillatory structure near the trailing edge of the pulse.