# 2.3.2 Higher-Order Nonlinear Effects

- Although the **propagation equation** (2.3.28) has been successful in explaining a large number of **nonlinear effects**, it may need to **be modified** depending on the **experimental conditions**.
  - For example, Eq.(2.3.28) does not include the effects of **stimulated inelastic scattering** such as **SRS** and **SBS**

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2}A = i\gamma(\omega_0)|A|^2 A, \qquad (2.3.28)$$

- If **peak power** of the **incident pulse** is above a **threshold level**, **SRS** and **SBS** can **transfer energy** to a new pulse at a **different wavelength**,
  - which may propagate in the **same** or the **opposite direction**.
  - The two pulses also interact with each other through the phenomenon of **cross-phase modulation (XPM).**

- A **similar situation** occurs when **two** or **more pulses** at **different wavelengths** (separated by more than **individual spectral widths**) are incident on the fiber.
- **Simultaneous propagation** of **multiple pulses** is governed by a set of equations similar to Eq. (2.3.28), modified suitably to include the contributions of **XPM** and the **Raman** or **Brillouin gain.**

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2}A = i\gamma(\omega_0)|A|^2 A, \qquad (2.3.28)$$

- Equation (2.3.28) should also be modified for **ultrashort optical pulses** whose widths are **close to** or **<1 ps**
  - The **spectral width** of such pulses becomes **large enough** that **several approximations** made in the derivation of Eq. (2.3.28) become **questionable**.
  - The **most important limitation** turns out to be the **neglect** of the **Raman effect**.

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2}A = i\gamma(\omega_0)|A|^2 A, \qquad (2.3.28)$$

  - **Intrapulse Raman scattering:** For pulses with a **wide spectrum (>0.1 THz)**, the **Raman gain** can amplify the **low-frequency components** of a pulse by **transferring energy** from the **high-frequency** components of the same pulse.
  - **Raman-induced frequency shift:** the **pulse spectrum** shifts toward the **low-frequency (red)** side as the pulse propagates inside the fiber.

- **The physical origin** of this effect is related to the **delayed nature** of the **Raman (vibrational) response**.
- The **starting point** is again the **wave equation** (2.3.1).

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}_L}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2}, \qquad (2.3.1)$$

- Equation (2.1.10) describes a **wide variety** of **third-order nonlinear effects**, and not all of them are **relevant to** our discussion.
    - For example, nonlinear phenomena such as **third-harmonic generation** and **four-wave mixing** are **unlikely to occur** unless an appropriate **phase-matching condition** is satisfied

$$\mathbf{P}_{NL}(\mathbf{r},t) = \varepsilon_0 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t} dt_3 \times \chi^{(3)}(t-t_1, t-t_2, t-t_3) \vdots \mathbf{E}(\mathbf{r},t_1)\mathbf{E}(\mathbf{r},t_2)\mathbf{E}(\mathbf{r},t_3). \qquad (2.1.10)$$

- The **intensity-dependent nonlinear effects** can be included by assuming the following form for the **third-order susceptibility** [16]:

$$\chi^{(3)}(t-t_1,t-t_2,t-t_3) = \chi^{(3)}R(t-t_1)\delta(t_1-t_2)\delta(t-t_3), \qquad (2.3.31)$$

$R(t)$: the **nonlinear response function** ( $\int_{-\infty}^{\infty} R(t)dt = 1$)

- If we **substitute** Eq. (2.3.31) in Eq. (2.1.10) and introduce a **slowly varying optical field** through Eq. (2.3.2), the **scalar form** of the **nonlinear polarization** is given by

$$P_{NL}(\mathbf{r},t) = \frac{3\varepsilon_0}{4}\chi^{(3)}_{xxxx}E(\mathbf{r},t)\int_{-\infty}^{t} R(t-t_1)E^*(\mathbf{r},t_1)E(\mathbf{r},t_1)\,dt_1, \qquad (2.3.32)$$

– the **upper limit** of **integration extends** only up to $t$
– the **response function**: $R(t-t_1)=0$ for $t_1 > t$ (**causality**)

$$\mathbf{P}_{NL}(\mathbf{r},t) = \varepsilon_0 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t} dt_3 \times \chi^{(3)}(t-t_1,t-t_2,t-t_3) \vdots \mathbf{E}(\mathbf{r},t_1)\mathbf{E}(\mathbf{r},t_2)\mathbf{E}(\mathbf{r},t_3). \qquad (2.1.10)$$

- In the **frequency domain**, $\tilde{E}$ is found to satisfy (using Eqs. (2.3.2)–(2.3.4))

$$\nabla^2\tilde{E} + n^2(\omega)k_0^2\tilde{E} = -ik_0\alpha - \chi_{xxxx}^{(3)}k_0^2 \iint_{-\infty}^{\infty} \tilde{R}(\omega_1-\omega_2) \times \tilde{E}(\omega_1,z)\tilde{E}^*(\omega_2,z)\tilde{E}(\omega-\omega_1+\omega_2,z)\,d\omega_1 d\omega_2, \qquad (2.3.33)$$

$\tilde{R}(\omega)$: the Fourier transform of $R(t)$.

- First, we can treat the terms on the **right-hand side** as a small **perturbation** and first obtain the **modal distribution** by **neglecting them**.
- The effect of **perturbation terms** is to change the **propagation constant** for the **fundamental mode** by $\Delta\beta$ as in Eq. (2.3.19) but with a **different expression** for $\Delta\beta$.

$$\tilde{\beta}(\omega) = \beta(\omega) + \Delta\beta(\omega), \qquad (2.3.19)$$

$$\mathbf{E}(\mathbf{r},t) = \tfrac{1}{2}\hat{x}[E(\mathbf{r},t)\exp(-i\omega_0 t) + \text{c.c.}], \qquad (2.3.2)$$

$$\mathbf{P}_L(\mathbf{r},t) = \tfrac{1}{2}\hat{x}[P_L(\mathbf{r},t)\exp(-i\omega_0 t) + \text{c.c.}], \qquad (2.3.3)$$

$$\mathbf{P}_{\text{NL}}(\mathbf{r},t) = \tfrac{1}{2}\hat{x}[P_{\text{NL}}(\mathbf{r},t)\exp(-i\omega_0 t) + \text{c.c.}]. \qquad (2.3.4)$$

- The **slowly varying amplitude** $A(z, t)$ can still be defined as in Eq. (2.3.21).

$$\mathbf{E}(\mathbf{r},t) = \tfrac{1}{2}\hat{x}\{F(x,y)A(z,t)\exp[i(\beta_0 z - \omega_0 t)] + \text{c.c.}\}, \qquad (2.3.21)$$

- When converting back from **frequency** to **time domain**, we should **take into account** the **frequency dependence** of $\Delta\beta$ by expanding it in a **Taylor series** as indicated in Eq. (2.3.25).

$$\Delta\beta(\omega) = \Delta\beta_0 + (\omega - \omega_0)\Delta\beta_1 + \tfrac{1}{2}(\omega - \omega_0)^2\Delta\beta_2 + \cdots, \qquad (2.3.25)$$

- This amounts to expanding the parameters $\gamma$ and $\alpha$ as

$$\gamma(\omega) = \gamma(\omega_0) + \gamma_1(\omega - \omega_0) + \tfrac{1}{2}\gamma_2(\omega - \omega_0)^2 + \cdots, \qquad (2.3.34)$$

$$\alpha(\omega) = \alpha(\omega_0) + \alpha_1(\omega - \omega_0) + \tfrac{1}{2}\alpha_2(\omega - \omega_0)^2 + \cdots, \qquad (2.3.35)$$

$$\gamma_m = (d^m\gamma/d\omega^m)_{\omega=\omega 0}$$

- In most cases of practical interest it is sufficient to retain the **first two terms** in this expansion

- We then obtain the following equation for **pulse evolution** inside a **single-mode fiber**:

$$\frac{\partial A}{\partial z}+\frac{1}{2}\left(\alpha(\omega_0)+i\alpha_1\frac{\partial}{\partial t}\right)A+\beta_1\frac{\partial A}{\partial t}+\frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2}-\frac{\beta_3}{6}\frac{\partial^3 A}{\partial t^3}$$
$$=i\left(\gamma(\omega_0)+i\gamma_1\frac{\partial}{\partial t}\right)\left(A(z,t)\int_0^\infty R(t')|A(z,t-t')|^2dt'\right). \qquad (2.3.36)$$

  - The **integral** in this equation accounts for the **energy transfer** resulting from **intrapulse Raman scattering**.

- Equation (2.3.36) can be used for pulses as short as a **few optical cycles** if enough **higher-order dispersive terms** are included .
  - For example, **dispersive effects** up to **12th order** are sometimes included when dealing with **supercontinuum generation** in **optical fibers**

$$\frac{\partial A}{\partial z}+\beta_1\frac{\partial A}{\partial t}+\frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2}+\frac{\alpha}{2}A=ir(\omega_0)|A|^2A, \qquad (2.3.28)$$

- It is important to note that the use of $\gamma_1$ in Eq. (2.3.36) includes automatically the **frequency dependence** of both $\boldsymbol{n_2}$ and $\boldsymbol{A_{eff}}$.
- The ratio $\gamma_1/\gamma$ consists of the following three terms

$$(\gamma_1 = (d\gamma/d\omega)_{\omega=\omega 0}, \quad \gamma = \frac{n_2(\omega_0)\omega_0}{cA_{eff}}):$$

$$\frac{\gamma_1(\omega_0)}{\gamma(\omega_0)} = \frac{1}{\omega_0} + \frac{1}{n_2}\left(\frac{dn_2}{d\omega}\right)_{\omega=\omega_0} - \frac{1}{A_{\text{eff}}}\left(\frac{dA_{\text{eff}}}{d\omega}\right)_{\omega=\omega_0}. \qquad (2.3.37)$$

  - The **first term** provides the **dominant contribution**,
  - The **second** and **third terms** become **important** in the case of a **supercontinuum** that may extend over **100 THz** or more
  - If **spectral broadening** is limited to **20 THz** or so, one can employ $\boldsymbol{\gamma_1 \approx \gamma/\omega_0}$
- If we combine the terms containing the **derivative** $\boldsymbol{\partial A/\partial t}$, we find that the $\gamma_1$ term forces the **group velocity** to depend on the **optical intensity** and leads to the phenomenon of **self-steepening**

- The nonlinear **response function** $R(t)$ should include both the **electronic** and **nuclear contributions**.

- Assuming that the **electronic contribution** is nearly **instantaneous**, the functional form of $R(t)$ can be written as

$$R(t) = (1 - f_R)\delta(t - t_e) + f_R h_R(t), \qquad (2.3.38)$$

  - ***$t_e$*** : a negligibly **short delay** in electronic response ($t_e$ < 1 fs).
  - ***$f_R$***: **fractional contribution** of the **delayed Raman response** to **nonlinear polarization** $P_{NL}$.
  - ***$h_R(t)$***：**Raman response function** (it is set by **vibrations** of **silica molecules** induced by the **optical field**)

- It is **not easy** to calculate $\boldsymbol{h_R(t)}$ because of the **amorphous nature** of **silica fibers**.
- An **indirect experimental approach** is **used in practice** by noting that the **Raman gain spectrum** is related to the **imaginary part** of the **Fourier transform** of $h_{R(t)}$ as

$$g_R(\Delta\omega) = \frac{\omega_0}{cn(\omega_0)} f_R \chi^{(3)}_{xxxx} \mathrm{Im}[\tilde{h}_R(\Delta\omega)], \qquad (2.3.39)$$

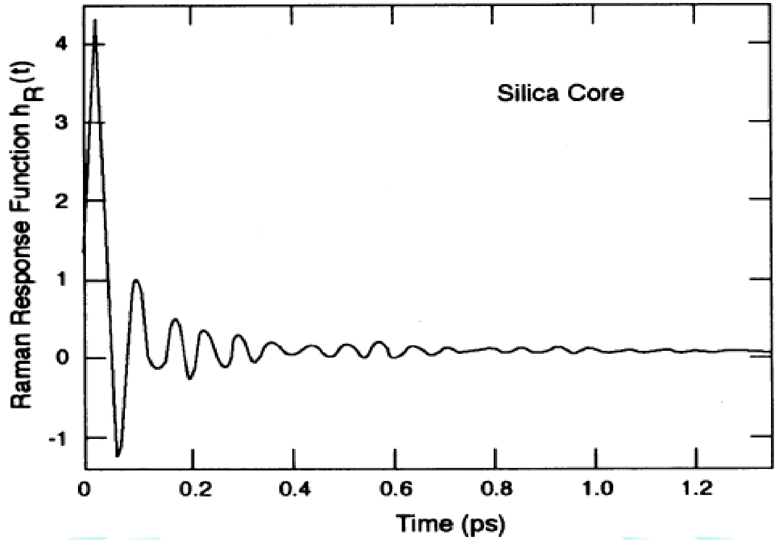
$\Delta\omega = \omega - \omega_0$

Im: stands for the **imaginary part**

- The **real part** of $\tilde{h}_R(\Delta\omega)$ can be obtained from the **imaginary part** through the **Kramers–Kronig relation**.
- The **inverse Fourier transform** of $\tilde{h}_R(\Delta\omega)$ then provides the **Raman response function** $h_R(t)$.

# Temporal variation of the Raman response function $h_R(t)$

- **Temporal form** of $h_R(t)$ deduced from the experimentally measured spectrum of the **Raman gain** in silica fibers [15].




- In view of the **damped oscillations**, a useful form is

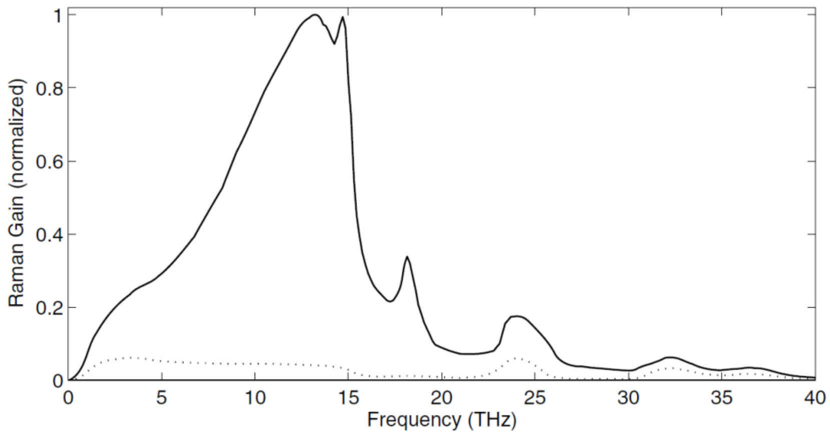
$$h_R(t) = \frac{\tau_1^2 + \tau_2^2}{\tau_1 \tau_2^2} \exp(-t/\tau_2) \sin(t/\tau_1). \qquad (2.3.40)$$

  – $\tau_1 = 12.2$ fs, $\tau_2 = 32$ fs

- The **fraction** $f_R$ can also be estimated from Eq. (2.3.39).

$$g_R(\Delta\omega) = \frac{\omega_0}{cn(\omega_0)} f_R \chi^{(3)}_{xxxx} \mathrm{Im}[\tilde{h}_R(\Delta\omega)], \qquad (2.3.39)$$

- Using the known numerical value of **peak Raman gain**, $f_R$ is found to be about **0.18**
- One should be careful in using Eq. (2.3.40) for $h_R(t)$ because it approximates the **actual Raman gain spectrum** with a **single Lorentzian profile**, and thus **fails to** reproduce the **hump** at frequencies below 5 THz.
- This feature contributes to the **Raman-induced frequency shift**. As a result, Eq. (2.3.40) generally underestimates this shift.




$$h_R(t) = \frac{\tau_1^2 + \tau_2^2}{\tau_1 \tau_2^2} \exp(-t/\tau_2)\sin(t/\tau_1)$$

(2.3.40)

- Equation (2.3.36) together with the **response function** *R*(*t*) given by Eq. (2.3.38) governs **evolution of ultrashort pulses** in **optical fibers**.

$$\frac{\partial A}{\partial z}+\frac{1}{2}\left(\alpha(\omega_0)+i\alpha_1\frac{\partial}{\partial t}\right)A+\beta_1\frac{\partial A}{\partial t}+\frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2}-\frac{\beta_3}{6}\frac{\partial^3 A}{\partial t^3}$$
$$=i\left(\gamma(\omega_0)+i\gamma_1\frac{\partial}{\partial t}\right)\left(A(z,t)\int_0^\infty R(t')|A(z,t-t')|^2dt'\right). \quad (2.3.36)$$

$$R(t)=(1-f_R)\delta(t-t_e)+f_R h_R(t), \quad (2.3.38)$$

- Its **accuracy** has been verified by showing that it **preserves** the **number of photons** during pulse evolution if **fiber loss is ignored** by setting ***α* = 0** [17].
- The **pulse energy** is **not conserved** in the presence of **intrapulse Raman scattering** because a part of the **pulse energy** is taken by **silica molecules**.
- Equation (2.3.36) includes this source of **nonlinear loss**.

- It is easy to see that it **reduces to** the **simpler equation** (2.3.28) for **optical pulses** much **longer** than the time scale of the **Raman response function** $h_R(t)$.
  - $h_R(t)$□ 0 for $\tau > 1$ ps,
  - for such pulses, $R(t)$ can be replaced by $\delta(t)$.

- As the **higher-order dispersion term** involving $\boldsymbol{\beta_3}$, the **loss term** involving $\boldsymbol{\alpha_1}$, and the **nonlinear term** involving $\boldsymbol{\gamma_1}$ are also **negligible** for such pulses, Eq. (2.3.36) reduces to Eq. (2.3.28).

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2}A = i\gamma(\omega_0)|A|^2 A, \qquad (2.3.28)$$

$$\frac{\partial A}{\partial z} + \frac{1}{2}\left(\alpha(\omega_0) + i\alpha_1 \frac{\partial}{\partial t}\right)A + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial t^3}$$
$$= i\left(\gamma(\omega_0) + i\gamma_1 \frac{\partial}{\partial t}\right)\left(A(z,t)\int_0^\infty R(t')|A(z,t-t')|^2 dt'\right). \qquad (2.3.36)$$

➢ For **pulses wide** enough to contain **many optical cycles** (pulse width >100 fs), we can simplify Eq. (2.3.36) considerably by setting $\boldsymbol{\alpha_1 = 0}$ and $\boldsymbol{\gamma_1 = \gamma/\omega_0}$ and using the following Taylor-series expansion:

$$|A(z,t-t')|^2 \approx |A(z,t)|^2 - t'\frac{\partial}{\partial t}|A(z,t)|^2. \tag{2.3.41}$$

➢ This approximation is **reasonable** if the **pulse envelope** evolves **slowly along the fiber**.

$$\frac{\partial A}{\partial z} + \frac{1}{2}\left(\alpha(\omega_0) + i\alpha_1\frac{\partial}{\partial t}\right)A + \beta_1\frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial t^3}$$
$$= i\left(\gamma(\omega_0) + i\gamma_1\frac{\partial}{\partial t}\right)\left(A(z,t)\int_0^\infty R(t')|A(z,t-t')|^2 dt'\right). \tag{2.3.36}$$

$$\frac{\gamma_1(\omega_0)}{\gamma(\omega_0)} = \frac{1}{\omega_0} + \frac{1}{n_2}\left(\frac{dn_2}{d\omega}\right)_{\omega=\omega_0} - \frac{1}{A_{\text{eff}}}\left(\frac{dA_{\text{eff}}}{d\omega}\right)_{\omega=\omega_0}. \tag{2.3.37}$$

$$R(t) = (1-f_R)\delta(t-t_e) + f_R h_R(t), \tag{2.3.38}$$

- Defining the **nonlinear response function** as

$$T_R \equiv \int_0^\infty tR(t)\,dt \approx f_R \int_0^\infty t\,h_R(t)\,dt = f_R \frac{d(\mathrm{Im}\,\tilde{h}_R)}{d(\Delta\omega)}\bigg|_{\Delta\omega=0} \qquad (2.3.42)$$

  - $\int_0^\infty R(t)\,dt = 1$ ,
  - Eq. (2.3.36) can be approximated by

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2}A + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} = i\gamma\left(|A|^2 A + \frac{i}{\omega_0}\frac{\partial}{\partial T}(|A|^2 A) - T_R A \frac{\partial |A|^2}{\partial T}\right) \qquad (2.3.43)$$

  - where a **frame** of **reference** moving with the pulse at the **group velocity** $v_g$ (the so-called **retarded frame**) is used

$$T = t - z/v_g \equiv t - \beta_1 z. \qquad (2.3.44)$$

- A **second-order term** involving the ratio $T_R / \omega_0$ was **neglected** in arriving at Eq. (2.3.43) because of its **smallness**.

$$R(t) = (1 - f_R)\delta(t - t_e) + f_R h_R(t), \qquad (2.3.38)$$

- It is easy to identify the origin of the **three higher-order terms**

$$\frac{\partial A}{\partial z}+\frac{\alpha}{2}A+\frac{i\beta_2}{2}\frac{\partial^2 A}{\partial T^2}-\frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3}=i\gamma\left(|A|^2A+\frac{i}{\omega_0}\frac{\partial}{\partial T}(|A|^2A)-T_RA\frac{\partial|A|^2}{\partial T}\right) \qquad (2.3.43)$$

  - The term proportional to $\boldsymbol{\beta_3}$ results from including the **cubic term** in the expansion of the **propagation constant**.
  - This term governs the effects of **third-order dispersion** and becomes important for **ultrashort pulses** because of their **wide bandwidth**.

$$\beta(\omega)=\beta_0+(\omega-\omega_0)\beta_1+\tfrac{1}{2}(\omega-\omega_0)^2\beta_2+\tfrac{1}{6}(\omega-\omega_0)^3\beta_3+\cdots,$$

$$\frac{\partial A}{\partial z}+\frac{\alpha}{2}A+\frac{i\beta_2}{2}\frac{\partial^2 A}{\partial T^2}-\frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3}=i\gamma\left(|A|^2A+\frac{i}{\omega_0}\frac{\partial}{\partial T}(|A|^2A)-T_RA\frac{\partial |A|^2}{\partial T}\right) \quad (2.3.43)$$

– The term proportional to $\omega_0^{-1}$ results from the **frequency dependence** of $\Delta\beta$ in Eq. (2.3.20). It is responsible for **self-steepening** .

$$\Delta\beta(\omega)=\frac{\omega^2 n(\omega)}{c^2\beta(\omega)}\frac{\iint_{-\infty}^{\infty}\Delta n(\omega)|F(x,y)|^2\,dxdy}{\iint_{-\infty}^{\infty}|F(x,y)|^2\,dxdy}. \quad (2.3.20)$$

– The **last term** proportional to $T_R$ has its origin in the **delayed Raman response**, and it is responsible for the **Raman-induced frequency shift** induced by **intrapulse Raman scattering**.

$$T_R \equiv \int_0^{\infty} tR(t)\,dt \approx f_R\int_0^{\infty} t\,h_R(t)\,dt = f_R\frac{d(\mathrm{Im}\,\tilde{h}_R)}{d(\Delta\omega)}\bigg|_{\Delta\omega=0} \quad (2.3.42)$$

- By using Eqs. (2.3.39) and (2.3.42), $T_R$ can be related to the **slope of the Raman gain spectrum** [12] that is assumed to vary **linearly** with **frequency** in the **vicinity** of the **carrier frequency** $\boldsymbol{\omega_0}$.

$$g_R(\Delta\omega) = \frac{\omega_0}{cn(\omega_0)} f_R \chi^{(3)}_{xxxx} \mathrm{Im}[\tilde{h}_R(\Delta\omega)], \qquad (2.3.39)$$

$$T_R \equiv \int_0^\infty tR(t)\,dt \approx f_R \int_0^\infty t\,h_R(t)\,dt = f_R \left.\frac{d(\mathrm{Im}\,\tilde{h}_R)}{d(\Delta\omega)}\right|_{\Delta\omega=0}, \qquad (2.3.42)$$

- Its **numerical value** has also been deduced experimentally, resulting in $T_R$=3 fs at wavelengths near **1.5 μm**.
- For pulses **shorter** than 0.5 ps, the **Raman gain** may **not** vary **linearly** over the entire pulse bandwidth, and the use of Eq. (2.3.43) becomes questionable for such short pulses.

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2}A + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} = i\gamma\left(|A|^2 A + \frac{i}{\omega_0}\frac{\partial}{\partial T}(|A|^2 A) - T_R A \frac{\partial |A|^2}{\partial T}\right) \qquad (2.3.43)$$

- For **pulses of width $T_0$ > 5 ps**, the parameters $(\omega_0 T_0)^{-1}$ and $T_R / T_0$ become so small ( < 0.001) that the last two terms in Eq. (2.3.43) can be neglected.

$$\frac{\partial A}{\partial z}+\frac{\alpha}{2}A+\frac{i\beta_2}{2}\frac{\partial^2 A}{\partial T^2}-\frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3}=i\gamma\left(|A|^2A+\frac{i}{\omega_0}\frac{\partial}{\partial T}(|A|^2A)-T_RA\frac{\partial|A|^2}{\partial T}\right) \qquad (2.3.43)$$

- As the contribution of the **third-order dispersion** term is also **quite small** for such pulses (as long as the carrier wavelength is not too close to the zero-dispersion wavelength), one can employ the simplified equation

$$i\frac{\partial A}{\partial z}+\frac{i\alpha}{2}A-\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2}+\gamma|A|^2A=0 \qquad (2.3.45)$$

- This equation can also be obtained from Eq. (2.3.28) by using the transformation given in Eq. (2.3.44).

$$\frac{\partial A}{\partial z}+\beta_1\frac{\partial A}{\partial t}+\frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2}+\frac{\alpha}{2}A=i\gamma(\omega_0)|A|^2A, \qquad (2.3.28)$$

$$T=t-z/v_g\equiv t-\beta_1 z. \qquad (2.3.44)$$

- Eq. (2.3.45) is referred to as the **NLS equation** (if **α = 0**) because it resembles the Schrödinger **with a nonlinear potential term** (variable z playing the role of time).

$$i\frac{\partial A}{\partial z}+\frac{i\alpha}{2}A-\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2}+\gamma|A|^2A=0. \qquad (2.3.45)$$

- Eq. (2.3.43) is called the **generalized (or extended) NLS equation**.

$$\frac{\partial A}{\partial z}+\frac{\alpha}{2}A+\frac{i\beta_2}{2}\frac{\partial^2 A}{\partial T^2}-\frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3}=i\gamma\left(|A|^2A+\frac{i}{\omega_0}\frac{\partial}{\partial T}(|A|^2A)-T_RA\frac{\partial|A|^2}{\partial T}\right) \qquad \textbf{(2.3.43)}$$

- The NLS equation is a **fundamental equation** of nonlinear science and has been studied extensively in the context of solitons
- Equation (2.3.45) is the **simplest** nonlinear equation for studying **third-order non-linear effects** in optical fibers

- If the **peak power** associated with an optical pulse becomes so large that one needs to include the **fifth** and **higher-order** terms in Eq. (1.3.1), the NLS equation needs to be modified.
-

$$\mathbf{P} = \varepsilon_0 \left( \chi^{(1)} \cdot \mathbf{E} + \chi^{(2)} : \mathbf{EE} + \chi^{(3)} \vdots \mathbf{EEE} + \cdots \right), \qquad (1.3.1)$$

- A **simple approach** replaces the **nonlinear parameter** with $\gamma$ governing the power level at which the nonlinearity begins to saturate
  - $\boldsymbol{\gamma = \gamma_0(1 - b_s\,|A|^2)}$ ,
  - $b_s$: a **saturation parameter**

- For **silica fibers,** $\boldsymbol{b_s |A|^2 \ll 1}$ in most practical situations, and one can use Eq. (2.3.45).

$$i\frac{\partial A}{\partial z} + \frac{i\alpha}{2}A - \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \gamma|A|^2 A = 0. \qquad (2.3.45)$$

- This term ($\gamma = \gamma_0(1 - b_s \mid A \mid^2)$ ) may **become relevant** when the peak intensity approaches **1 GW/cm²** .
  - The resulting equation is called the **cubic-quintic NLS equation**
  - it contains terms involving both the **third** and **fifth powers** of the amplitude A.
- Eq. (2.3.45) is referred to as the cubic NLS equation.

$$i\frac{\partial A}{\partial z} + \frac{i\alpha}{2}A - \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \gamma|A|^2 A = 0 \qquad (2.3.45)$$

- Fibers made by using materials with **larger values of $n_2$** (such as **silicate** and **chalcogenide fibers(硫化物)**) are likely to exhibit the **saturation effects** at a lower peak-power level.

- The **cubic-quintic NLS** equation may become relevant for them and for fibers whose **core** is doped with **high-nonlinearity materials** such as **organic dyes** [42] and **semiconductors** [43].

- Equation (2.3.45) appears in **optics** in several **different contexts** [39].

$$i\frac{\partial A}{\partial z} + \frac{i\alpha}{2}A - \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \gamma|A|^2 A = 0 \qquad (2.3.45)$$

  - For example, the same equation holds for propagation of CW beams in **planar waveguides** when the variable ***T*** is interpreted as a **spatial coordinate**.
  - The $\beta_2$ term in Eq. (2.3.45) then governs **beam diffraction** in the **plane of the waveguide**.

- This analogy between “**diffraction in space**” and “**dispersion in time**” is often exploited to advantage since the same equation governs the underlying physics.

# 2.4 Numerical Methods

- The NLS equation is a **nonlinear partial differential equation** that does not generally lend itself to **analytic solutions** except for some specific cases in which the **inverse scattering method (逆散射)** can be employed
- **A numerical approach** is therefore often necessary for an understanding of **the nonlinear effects** in optical fibers.
- **Numerical methods** can be classified into two broad categories
  1. Finite-difference method (有限差分法):
  2. Pseudo-spectral method (偽譜法): are faster by up to **an order of magnitude** to achieve the **same accuracy**
- **Split-step Fourier method (分步傅立葉法):**
  - The one method that has been used extensively to solve the pulse-propagation problem in **nonlinear dispersive media** .
  - **The relative Speed compare with Finite-difference** method can be attributed in part to the use of the **finite-Fourier-transform (FFT) algorithm** .

# Split-Step Fourier Method

- To understand the philosophy behind the **split-step Fourier method**, it is useful to write Eq. (2.3.43) formally in the form

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2}A + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} = i\gamma\left(|A|^2 A + \frac{i}{\omega_0}\frac{\partial}{\partial T}(|A|^2 A) - T_R A\frac{\partial |A|^2}{\partial T}\right) \quad (2.3.43)$$

$$\frac{\partial A}{\partial z} = (\hat{D} + \hat{N})A, \quad (2.4.1)$$

$$\hat{D} = -\frac{i\beta_2}{2}\frac{\partial^2}{\partial T^2} + \frac{\beta_3}{6}\frac{\partial^3}{\partial T^3} - \frac{\alpha}{2}, \quad (2.4.2)$$

$$\hat{N} = i\gamma\left(|A|^2 + \frac{i}{\omega_0}\frac{1}{A}\frac{\partial}{\partial T}(|A|^2 A) - T_R\frac{\partial |A|^2}{\partial T}\right). \quad (2.4.3)$$

$\hat{D}$ (differential operator): accounts for **dispersion** and **losses** within a **linear medium**

$\hat{N}$ (nonlinear operator): the effect of **fiber nonlinearities** on pulse propagation.

- **Dispersion** and **nonlinearity** act together along the length of the fiber.

- **The split-step Fourier method** obtains an **approximate solution** by assuming that in propagating the optical field over **a small distance *h***,
  - **dispersive** and **nonlinear effects** can be assumed to act **independently**.
  - propagation from ***z*** to ***z+h*** is carried out in two steps.
    - First step, the **nonlinearity** acts alone, and $\hat{D}=0$
    - Second step, **dispersion** acts alone, and $\hat{N}=0$
  - **Mathematically**

$$A(z+h,T) \approx \exp(h\hat{D})\exp(h\hat{N})A(z,T). \qquad (2.4.4)$$

- The **exponential operator** $\exp(h\hat{D})$ can be evaluated in the **Fourier domain** using the prescription

$$\exp(h\hat{D})B(z,T) = F_T^{-1}\exp[h\hat{D}(-i\omega)]F_T B(z,T), \qquad (2.4.5)$$

$F_T$: the **Fourier-transform operation**

$\hat{D}(-i\omega)$ : from Eq. (2.4.2) by replacing the **operator** $\frac{\partial}{\partial T}$ by -iω,

**ω**: frequency in the Fourier domain.

- As $\hat{D}$ (i$\omega$) is just **a number in the Fourier space**, the evaluation of Eq. (2.4.5) is **straightforward**.
- The use of the **FFT algorithm** makes **numerical evaluation of Eq. (2.4.5)** relatively fast.
- It is for this reason that the **split-step Fourier method** can be faster by **up to two orders of magnitude** compared with most **finite-difference schemes.**

- To estimate the accuracy of **the split-step Fourier method**, that a formally **exact solution** of Eq. (2.4.1) is given by ($\hat{N}$ is assumed to be **z independent**)

$$\frac{\partial A}{\partial z} = (\hat{D} + \hat{N})A, \quad (2.4.1)$$

$$A(z+h,T) = \exp[h(\hat{D}+\hat{N})]A(z,T), \quad (2.4.6)$$

- The **Baker– Hausdorff (貝克豪斯多夫) formula** for two noncommuting operators $\hat{a}$ and $\hat{b}$ ($[\hat{a}, \hat{b}] = \hat{a}\hat{b} - \hat{b}\hat{a}$)

$$\exp(\hat{a})\exp(\hat{b}) = \exp\left(\hat{a}+\hat{b}+\frac{1}{2}[\hat{a},\hat{b}]+\frac{1}{12}[\hat{a}-\hat{b},[\hat{a},\hat{b}]]+\cdots\right), \quad (2.4.7)$$

- A comparison of Eqs. (2.4.4) and (2.4.6) shows that the **split-step Fourier method ignores the noncommutating** nature of the operators $\hat{D}$ and $\hat{N}$ .

$$A(z+h,T) \approx \exp(h\hat{D})\exp(h\hat{N})A(z,T). \quad (2.4.4)$$

- By using Eq. (2.4.7) with $\hat{a} = h\,\widehat{D}$ and $\hat{b} = h\,\widehat{N}$, the **dominant error** term is found to result from the commutator $\frac{1}{2}h^2[\widehat{D}\,,\widehat{N}\,]$.

$$A(z+h,T) = \exp[h(\hat{D}+\hat{N})]A(z,T), \qquad (2.4.6)$$

- The split-step Fourier method is accurate to **second order** in the step size $h$

$$\exp(\hat{a})\exp(\hat{b}) = \exp\left(\hat{a}+\hat{b}+\frac{1}{2}[\hat{a},\hat{b}]+\frac{1}{12}[\hat{a}-\hat{b},[\hat{a},\hat{b}]]+\cdots\right), \qquad (2.4.7)$$

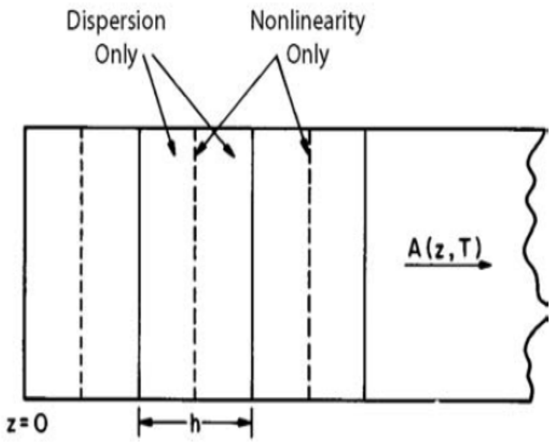
$$A(z+h,T) \approx \exp(h\hat{D})\exp(h\hat{N})A(z,T). \qquad (2.4.4)$$

- The **accuracy** of the **split-step Fourier method** can be improved by adopting a **different procedure** to propagate the **optical pulse** over one segment from *z* to *z*+*h*.
- In this **procedure** Eq. (2.4.4) is replaced by

$$A(z+h,T) \approx \exp(h\hat{D})\exp(h\hat{N})A(z,T). \qquad (2.4.4)$$

$$A(z+h,T) \approx \exp\left(\frac{h}{2}\hat{D}\right)\exp\left(\int_z^{z+h}\hat{N}(z')\,dz'\right)\exp\left(\frac{h}{2}\hat{D}\right)A(z,T). \qquad (2.4.8)$$

- The **main difference** is that the **effect of nonlinearity** is included **in the middle of the segment** rather than at the segment boundary.



- **Symmetrized split step Fourier method**:
  - the **symmetric** form of the **exponential operators** in Eq. (2.4.8)

$$A(z+h,T) \approx \exp\left(\frac{h}{2}\hat{D}\right)\exp\left(\int_z^{z+h}\hat{N}(z')\,dz'\right)\exp\left(\frac{h}{2}\hat{D}\right)A(z,T). \qquad (2.4.8)$$

- The integral in the **middle exponential** is useful to include the **z dependence** of the nonlinear operator $\widehat{N}$.

- If step size ($h$) is **small enough**, the **middle exponential** can be approximated by **exp ($h\widehat{N}$)**

- The most important **advantage** of using the **symmetrized form** of Eq. (2.4.8):

  leading **error term** results from the **double commutator** in Eq. (2.4.7) (**third order** in the step size $h$)

- This can be **verified** by applying Eq. (2.4.7) twice in Eq. (2.4.8)

$$\exp(\hat{a})\exp(\hat{b}) = \exp\left(\hat{a}+\hat{b}+\frac{1}{2}[\hat{a},\hat{b}]+\frac{1}{12}[\hat{a}-\hat{b},[\hat{a},\hat{b}]]+\cdots\right), \qquad (2.4.7)$$

- The accuracy of the **split-step Fourier method** can be further improved by evaluating the **integral** in Eq. (2.4.8) more accurately than approximating it by $h\,\hat{N}\,(z)$.

$$A(z+h,T) \approx \exp\left(\frac{h}{2}\hat{D}\right)\exp\left(\int_{z}^{z+h}\hat{N}(z')\,dz'\right)\exp\left(\frac{h}{2}\hat{D}\right)A(z,T). \qquad (2.4.8)$$

- A simple approach is to employ the **trapezoidal rule (梯形法則)** and **approximate the integral** by

$$\int_{z}^{z+h} \hat{N}(z')\,dz' \approx \frac{h}{2}[\hat{N}(z)+\hat{N}(z+h)]. \quad (2.4.9)$$

  - $\hat{N}$ (z+h) is **unknown** at the mid-segment located at z+h/2.
  - It is **necessary** to follow an **iterative procedure**
    - it is initiated by replacing $\hat{N}$(z+h) by $\hat{N}$**(**z**)**.
    - Equation (2.4.8) is then used to estimate $A$(z+$h$,$T$)
    - it is used to calculate the new value of $\hat{N}$(z+h).

- It can still reduce the overall computing time if the **step size *h* can be increased** because of the improved accuracy of the **numerical algorithm**.

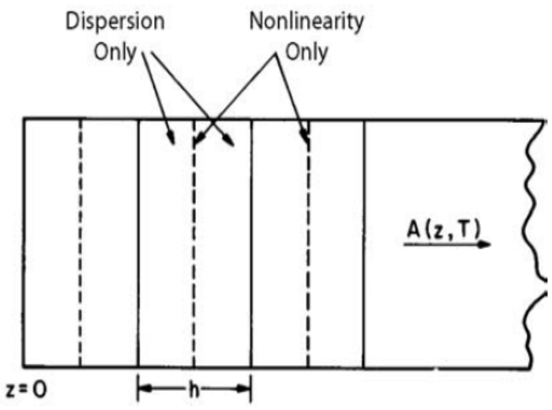
$$\hat{N} = i\gamma\left(|A|^2 + \frac{i}{\omega_0}\frac{1}{A}\frac{\partial}{\partial T}(|A|^2A) - T_R\frac{\partial |A|^2}{\partial T}\right). \quad (2.4.3)$$

- The implementation of the **split-step Fourier method** is relatively straightforward.
- The fiber length is divided into a **large number** of **segments** that need not be **spaced equally**.
- The **optical pulse** is propagated from segment to segment using the prescription of Eq. (2.4.8).

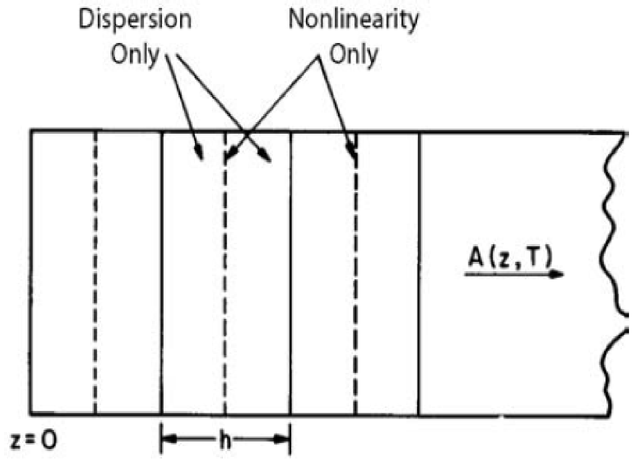
$$A(z+h,T) \approx \exp\left(\frac{h}{2}\hat{D}\right)\exp\left(\int_z^{z+h}\hat{N}(z')\,dz'\right)\exp\left(\frac{h}{2}\hat{D}\right)A(z,T). \qquad (2.4.8)$$

- The **optical field** *A*(z,T) is first propagated for a distance h/2 with **dispersion only** using the **FFT algorithm** and Eq. (2.4.5).

$$\exp(h\hat{D})B(z,T) = F_T^{-1}\exp[h\hat{D}(-i\omega)]F_T B(z,T), \qquad (2.4.5)$$



- At the **midplane** z+***h***/2, the field is multiplied by a **nonlinear term** that represents the effect of **nonlinearity** over the whole segment length ***h***.
- Finally, the field is propagated for the remaining distance ***h***/2 with **dispersion** only to obtain $A(z+\boldsymbol{h},T)$.
- In effect, the **nonlinearity** is assumed to be **lumped** at the **mid-plane** of each segment(dashed lines in Figure 2.3).



- The **split-step Fourier method** can be made to run faster by noting that the application of Eq. (2.4.8) over **M successive steps** results in the following expression:

$$A(L,T) \approx e^{-\frac{1}{2}h\hat{D}} \left( \prod_{m=1}^{M} e^{h\hat{D}} e^{h\hat{N}} \right) e^{\frac{1}{2}h\hat{D}} A(0,T). \qquad (2.4.10)$$

$L = \mathrm{M}h$: total fiber length

The integral in Eq. (2.4.9) was approximated with $h\widehat{N}$ .

- Thus, **except for the first** and **last dispersive steps**, **all intermediate steps** can be carried over the whole segment length ***h***.

- This feature **reduces** the required number of FFTs roughly by a factor of 2 and **speeds up** the **numerical code** by the same factor.

- A different algorithm is obtained if we use Eq. (2.4.7) with $\hat{a} = h\hat{N}$ and $\hat{b} = \hat{h}D$. In that case, Eq. (2.4.10) is replaced with

$$A(L,T) \approx e^{-\frac{1}{2}h\hat{N}} \left( \prod_{m=1}^{M} e^{h\hat{N}} e^{h\hat{D}} \right) e^{\frac{1}{2}h\hat{N}} A(0,T). \qquad (2.4.11)$$

- Both of these algorithms provide the **same accuracy** and are easy to implement in practice (see Appendix B).

- **Higher-order versions** of the **split-step Fourier method** can be used to improve the **computational efficiency**.

- The use of an **adaptive step size** along *z* can also help in **reducing the computational time** for certain problems .

- The **split-step Fourier method** has been applied to a **wide variety** of optical problems :
  1. wave propagation in atmosphere
  2. graded-index fibers (漸變折射率光纖)
  3. semiconductor lasers
  4. unstable resonators
  5. waveguide couplers

- **Beam-propagation method (光束傳輸法):**
  – when applied to the propagation of **CW optical beams** in **nonlinear media** when dispersion is replaced by diffraction.

# Non-linear Schrodinger Equation

- https://www.youtube.com/watch?v=5vw0Csy_1rs&t=143s