

## 2.3 Pulse-Propagation Equation

- The study of most nonlinear effects in optical fibers involves the use of **short pulses** with widths ranging from **~10 ns** to **10 fs**.
- Both **dispersive** and **nonlinear effects** influence their **shapes** and **spectra**.
- By using 2.1 Maxwell's Equations, it can be written in the form

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}_L}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2}, \quad (2.3.1)$$

$$\nabla \times \nabla \times \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_L(\mathbf{r}, t) + \mathbf{P}_{NL}(\mathbf{r}, t)$$

$$\nabla \times \nabla \times \mathbf{E} \equiv \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$$



$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}_L}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2}$$

- The **electric field** can be written in the form by **slowly varying envelope approximation** (separate the **rapidly varying part**)

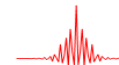
$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} \hat{x} [E(\mathbf{r}, t) \exp(-i\omega_0 t) + \text{c.c.}], \quad (2.3.2)$$

- $E(\mathbf{r}, t)$  : slowly varying function of time (relative to the optical period).

- The **polarization components**  $\mathbf{P}_L$  and  $\mathbf{P}_{NL}$  can also be expressed in a similar way by writing

$$\mathbf{P}_L(\mathbf{r}, t) = \frac{1}{2} \hat{x} [P_L(\mathbf{r}, t) \exp(-i\omega_0 t) + \text{c.c.}], \quad (2.3.3)$$

$$\mathbf{P}_{NL}(\mathbf{r}, t) = \frac{1}{2} \hat{x} [P_{NL}(\mathbf{r}, t) \exp(-i\omega_0 t) + \text{c.c.}]. \quad (2.3.4)$$



- The **linear component**  $P_L$  can be obtained by substituting **Eq. (2.3.3)** in **(2.1.9)**

$$\begin{aligned} P_L(\mathbf{r}, t) &= \epsilon_0 \int_{-\infty}^{\infty} \chi_{xx}^{(1)}(t - t') E(\mathbf{r}, t') \exp[i\omega_0(t - t')] dt' \\ &= \frac{\epsilon_0}{2\pi} \int_{-\infty}^{\infty} \tilde{\chi}_{xx}^{(1)}(\omega) \tilde{E}(\mathbf{r}, \omega - \omega_0) \exp[-i(\omega - \omega_0)t] d\omega, \end{aligned} \quad (2.3.5)$$

$\tilde{E}(\mathbf{r}, \omega)$ : Fourier transform of  $E(\mathbf{r}, t)$

$$\tilde{E}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} E(\mathbf{r}, t) \exp(i\omega t) dt. \quad (2.1.12)$$

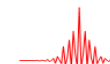
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$$\mathbf{P}_L(\mathbf{r}, t) = \frac{1}{2} \hat{x} [P_L(\mathbf{r}, t) \exp(-i\omega_0 t) + \text{c.c.}], \quad (2.3.3)$$

$$\mathbf{P}_L(\mathbf{r}, t) = \epsilon_0 \int_{-\infty}^t \chi^{(1)}(t - t') \cdot \mathbf{E}(\mathbf{r}, t') dt', \quad (2.1.9)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} \hat{x} [E(\mathbf{r}, t) \exp(-i\omega_0 t) + \text{c.c.}], \quad (2.3.2)$$

In considering Eq. (2.1.9) and Eq. (2.3.2)



- The **nonlinear component**  $P_{\text{NL}}(\mathbf{r}, t)$  is obtained by **substituting** Eq. (2.3.4) in Eq. (2.1.10)

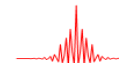
$$\mathbf{P}_{\text{NL}}(\mathbf{r}, t) = \frac{1}{2} \hat{x} [P_{\text{NL}}(\mathbf{r}, t) \exp(-i\omega_0 t) + \text{c.c.}]. \quad (2.3.4)$$

$$\begin{aligned} \mathbf{P}_{\text{NL}}(\mathbf{r}, t) = \epsilon_0 \int_{-\infty}^t dt_1 \int_{-\infty}^t dt_2 \int_{-\infty}^t dt_3 \\ \times \chi^{(3)}(t-t_1, t-t_2, t-t_3) : \mathbf{E}(\mathbf{r}, t_1) \mathbf{E}(\mathbf{r}, t_2) \mathbf{E}(\mathbf{r}, t_3). \end{aligned} \quad (2.1.10)$$

- The **time dependence** of  $\chi^{(3)}$  in Eq. (2.1.10) is given by the product of three **delta functions** of the form  $\delta(t-t_1)$  ( if the **nonlinear response** is assumed to be **instantaneous**)

$$\mathbf{P}_{\text{NL}}(\mathbf{r}, t) = \epsilon_0 \chi^{(3)} : \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t). \quad (2.3.6)$$

- The assumption of **instantaneous nonlinear response** amounts to **neglecting** the contribution of **molecular vibrations** to  $\chi^{(3)}$  (the **Raman effect**).



- Both **electrons** and **nuclei** respond to the **optical field** in a **nonlinear manner**.
- The **nuclei response** is **inherently slower** compared with the **electronic response**.
- For **silica fibers**, the **vibrational** or **Raman response** occurs over a time scale **60–70 fs**.
- Thus, Eq. (2.3.6) is approximately valid for **pulse widths**  $> 1$  ps.

$$\mathbf{P}_{\text{NL}}(\mathbf{r}, t) = \epsilon_0 \chi^{(3)} : \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t). \quad (2.3.6)$$

- When **Eq. (2.3.2)** is substituted in **Eq. (2.3.6)**,  $\mathbf{P}_{\text{NL}}(\mathbf{r}, t)$  is found to have a term **oscillating at  $\omega_0$**  and another term oscillating at the **third-harmonic frequency  $3\omega_0$** .

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} \hat{x} [E(\mathbf{r}, t) \exp(-i\omega_0 t) + \text{c.c.}], \quad (2.3.2)$$

$$\mathbf{P}_{\text{NL}}(\mathbf{r}, t) = \epsilon_0 \chi^{(3)} : \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t). \quad (2.3.6)$$

- The **latter term** requires **phase matching** and is generally **negligible** in optical fibers.

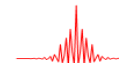


- To obtain the **wave equation** for the **slowly varying amplitude**  $E(\mathbf{r}, t)$ , it is **more convenient** to work in the **Fourier domain**.
- This is generally **not possible** as Eq. (2.3.1) is **nonlinear** because of the **intensity dependence** of  $\epsilon_{\text{NL}}$ .

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}_L}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}_{\text{NL}}}{\partial t^2}, \quad (2.3.1)$$

- In one **approach**,  $\epsilon_{\text{NL}}$  is treated as a **constant** during the derivation of the **propagation equation**.
- The **approach** is **justified** in view of the
  - **slowly varying envelope approximation**
  - **perturbative nature** of  $P_{\text{NL}}$ .

$$\epsilon_{\text{NL}} = \frac{3}{4} \chi_{\text{xxxx}}^{(3)} |E(\mathbf{r}, t)|^2. \quad (2.3.8)$$





- Substituting Eqs. (2.3.2) through (2.3.4) in Eq. (2.3.1), the **Fourier transform**  $\tilde{E}(\mathbf{r}, \omega - \omega_0)$ , defined as

$$\tilde{E}(\mathbf{r}, \omega - \omega_0) = \int_{-\infty}^{\infty} E(\mathbf{r}, t) \exp[i(\omega - \omega_0)t] dt, \quad (2.3.9)$$

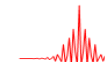
is found to satisfy the **Helmholtz equation** ( $k_0 = \omega/c$ )

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$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} \hat{x} [E(\mathbf{r}, t) \exp(-i\omega_0 t) + \text{c.c.}], \quad (2.3.2)$$

$$\mathbf{P}_{\text{NL}}(\mathbf{r}, t) = \frac{1}{2} \hat{x} [P_{\text{NL}}(\mathbf{r}, t) \exp(-i\omega_0 t) + \text{c.c.}]. \quad (2.3.4)$$

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}_L}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}_{\text{NL}}}{\partial t^2}, \quad (2.3.1)$$



- Substituting Eqs. (2.3.2) through (2.3.4) in Eq. (2.3.1), the **Fourier transform**  $\tilde{E}(\mathbf{r}, \omega - \omega_0)$ , defined as

$$\tilde{E}(\mathbf{r}, \omega - \omega_0) = \int_{-\infty}^{\infty} E(\mathbf{r}, t) \exp[i(\omega - \omega_0)t] dt, \quad (2.3.9)$$

is found to satisfy the **Helmholtz equation** ( $k_0 = \omega/c$ )

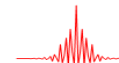
$$\nabla^2 \tilde{E} + \epsilon(\omega) k_0^2 \tilde{E} = 0, \quad (2.3.10)$$

the **dielectric constant**

$$\epsilon(\omega) = 1 + \tilde{\chi}_{xx}^{(1)}(\omega) + \epsilon_{\text{NL}} \quad (2.3.11)$$

**nonlinear part**  $\epsilon_{\text{NL}}$

$$\epsilon_{\text{NL}} = \frac{3}{4} \chi_{xxxx}^{(3)} |E(\mathbf{r}, t)|^2. \quad (2.3.8)$$



- The **dielectric constant** can be used to define the **refractive index**  $\tilde{n}$  and the **absorption coefficient**  $\tilde{\alpha}$ .
- Both  $\tilde{n}$  and  $\tilde{\alpha}$  become **intensity dependent** because of  $\epsilon_{\text{NL}}$ . It is customary to introduce

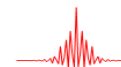
$$\tilde{n} = n + n_2 |E|^2, \quad \tilde{\alpha} = \alpha + \alpha_2 |E|^2 \quad (2.3.12)$$

- Using  $\epsilon = (\tilde{n} + i \tilde{\alpha} / 2k_0)^2$  and Eqs. (2.3.8) and (2.3.11), the **nonlinear-index coefficient**  $n_2$  and the **two-photon absorption coefficient**  $\alpha_2$  are given by

$$n_2 = \frac{3}{8n} \text{Re}(\chi_{xxxx}^{(3)}), \quad \alpha_2 = \frac{3\omega_0}{4nc} \text{Im}(\chi_{xxxx}^{(3)}). \quad (2.3.13)$$

$$\epsilon_{\text{NL}} = \frac{3}{4} \chi_{xxxx}^{(3)} |E(\mathbf{r}, t)|^2. \quad (2.3.8)$$

$$\epsilon(\omega) = 1 + \tilde{\chi}_{xx}^{(1)}(\omega) + \epsilon_{\text{NL}} \quad (2.3.11)$$



- The **linear index**  $n$  and the **absorption coefficient**  $\alpha$  are related to the **real** and **imaginary parts** of  $\tilde{\chi}_{xx}^{(1)}$  as in Eqs. (2.1.15) and (2.1.16).

$$n(\omega) = 1 + \frac{1}{2} \text{Re}[\tilde{\chi}^{(1)}(\omega)], \quad (2.1.15)$$

$$\alpha(\omega) = \frac{\omega}{nc} \text{Im}[\tilde{\chi}^{(1)}(\omega)], \quad (2.1.16)$$

- As  $\alpha_2$  is **relatively small** for silica fibers, it is often **ignored**.

- Equation (2.3.10) (**Helmholtz equation**) can be solved by using the method of **separation of variables**. If we assume a solution of the form

$$\tilde{E}(r, \omega - \omega_0) = F(x, y) \tilde{A}(z, \omega - \omega_0) \exp(i\beta_0 z), \quad (2.3.14)$$

$\tilde{A}(\mathbf{z}, \omega)$ : slowly varying function of  $\mathbf{z}$

$\beta_0$ : wave number

- Helmholtz Eq. (2.3.10) leads to the following two equations for  $F(x, y)$  and  $\tilde{A}(z, \omega)$ :

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + [\varepsilon(\omega)k_0^2 - \tilde{\beta}^2]F = 0, \quad (2.3.15)$$

$$2i\beta_0 \frac{\partial \tilde{A}}{\partial z} + (\tilde{\beta}^2 - \beta_0^2)\tilde{A} = 0 \quad (2.3.16)$$

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$$\nabla^2 \tilde{E} + \varepsilon(\omega)k_0^2 \tilde{E} = 0, \quad (2.3.10)$$

- In obtaining **Eq. (2.3.16)**, the **second derivative**  $\partial^2 \tilde{A} / \partial z^2$  was neglected since  $\tilde{A}(\mathbf{z}, \omega)$  is assumed to be a **slowly varying function** of  $\mathbf{z}$ .

$$2i\beta_0 \frac{\partial \tilde{A}}{\partial z} + (\tilde{\beta}^2 - \beta_0^2) \tilde{A} = 0. \quad (2.3.16)$$

- The **wave number**  $\tilde{\beta}$  is determined by solving the **eigenvalue equation (2.3.15)** for the **fiber modes** using a procedure similar to that used in Section 2.2.

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + [\varepsilon(\omega)k_0^2 - \tilde{\beta}^2]F = 0, \quad (2.3.15)$$

- The **dielectric constant**  $\varepsilon(\omega)$  in **Eq. (2.3.15)** can be approximated by

$$\varepsilon = (n + \Delta n)^2 \approx n^2 + 2n\Delta n, \quad (2.3.17)$$

- $\Delta n$  : small perturbation

$$\Delta n = n_2 |E|^2 + \frac{i\tilde{\alpha}}{2k_0}. \quad (2.3.18)$$

- **Eq. (2.3.15)** can be solved using **first-order perturbation theory**. We first replace  $\varepsilon$  with  $n^2$  and obtain the modal distribution  $F(x,y)$ , and the corresponding wave number  $\beta(\omega)$ .

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + [\varepsilon(\omega)k_0^2 - \tilde{\beta}^2]F = 0, \quad (2.3.15)$$



- For a **single-mode fiber**,  $F(x,y)$  corresponds to the **modal distribution** of the fundamental fiber mode  $\mathbf{HE}_{11}$  given by Eqs. (2.2.12) and (2.2.13), or by the **Gaussian approximation** (2.2.14).

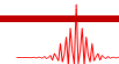
$$F(x, y) \approx \exp[-(x^2 + y^2)/w^2], \quad (2.2.14)$$

- We then include the effect of  $\Delta n$  in Eq. (2.3.15).
- In the **first-order perturbation theory**,  $\Delta n$  does not affect the **modal distribution**  $F(x,y)$ .
- The **eigenvalue**  $\tilde{\beta}$  becomes

$$\tilde{\beta}(\omega) = \beta(\omega) + \Delta\beta(\omega), \quad (2.3.19)$$

Where

$$\Delta\beta(\omega) = \frac{\omega^2 n(\omega)}{c^2 \beta(\omega)} \frac{\iint_{-\infty}^{\infty} \Delta n(\omega) |F(x, y)|^2 dx dy}{\iint_{-\infty}^{\infty} |F(x, y)|^2 dx dy} \quad (2.3.20)$$



- This step completes the **formal solution** of **Eq. (2.3.1)** to the first order in perturbation  $\mathbf{P}_{\text{NL}}$ .

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}_L}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}_{\text{NL}}}{\partial t^2}, \quad (2.3.1)$$

- Using **Eqs. (2.3.2)** and **(2.3.14)**, the **electric field  $\mathbf{E}(\mathbf{r}, t)$**  can be written as :

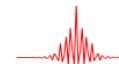
$$E(r, t) = \frac{1}{2} \hat{x} \{ F(x, y) A(z, t) \exp[i(\beta_0 z - \omega_0 t)] + \text{c.c.} \}, \quad (2.3.21)$$

**$\mathbf{A}(z, t)$  : the slowly varying pulse envelope**

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$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} \hat{x} [E(\mathbf{r}, t) \exp(-i\omega_0 t) + \text{c.c.}], \quad (2.3.2)$$

$$\tilde{E}(\mathbf{r}, \omega - \omega_0) = F(x, y) \tilde{A}(z, \omega - \omega_0) \exp(i\beta_0 z), \quad (2.3.14)$$



- The **Fourier transform**  $\tilde{A}(z, \omega - \omega_0)$  of  $A(z, t)$  satisfies **Eq. (2.3.16)**, which can be written as

$$\frac{\partial \tilde{A}}{\partial z} = i[\beta(\omega) + \Delta\beta(\omega) - \beta_0]\tilde{A}, \quad (2.3.22)$$

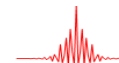
we used **Eq. (2.3.19)**  $\tilde{\beta}(\omega) = \beta(\omega) + \Delta\beta(\omega)$ ,

and approximated  $\tilde{\beta}^2 - \beta_0^2$  by  $2\beta_0(\tilde{\beta} - \beta_0)$ .

- The **physical meaning** of this equation is clear.
- Each **spectral component** within the **pulse envelope** acquires a **phase shift** whose magnitude is both **frequency** and **intensity dependent** as it propagates down the fiber, .

$$2i\beta_0 \frac{\partial \tilde{A}}{\partial z} + (\tilde{\beta}^2 - \beta_0^2)\tilde{A} = 0. \quad (2.3.16)$$

$$\tilde{\beta}(\omega) = \beta(\omega) + \Delta\beta(\omega), \quad (2.3.19)$$



- At this point, we can go back to the **time domain** by taking the **inverse Fourier transform** of **Eq. (2.3.22)**, and obtain the propagation equation for  $\mathbf{A}(z, t)$ .
- As an **exact functional form** of  $\beta(\omega)$  is rarely known, it is useful to expand  $\beta(\omega)$  in a **Taylor series** around the carrier frequency  $\omega_0$  as

$$\beta(\omega) = \beta_0 + (\omega - \omega_0)\beta_1 + \frac{1}{2}(\omega - \omega_0)^2\beta_2 + \frac{1}{6}(\omega - \omega_0)^3\beta_3 + \dots, \quad (2.3.23)$$

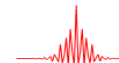
where  $\beta_0 \equiv \beta(\omega_0)$  and other parameters are defined as

$$\beta_m = \left( \frac{d^m \beta}{d\omega^m} \right)_{\omega=\omega_0} \quad (m = 1, 2, \dots). \quad (2.3.24)$$

- A similar expansion should be made for  $\Delta\beta(\omega)$ , i.e.,

$$\Delta\beta(\omega) = \Delta\beta_0 + (\omega - \omega_0)\Delta\beta_1 + \frac{1}{2}(\omega - \omega_0)^2\Delta\beta_2 + \dots, \quad (2.3.25)$$

$\Delta\beta_m$  is defined similar to **Eq. (2.3.24)**.



- The **cubic** and **higher-order terms** in the expansion (2.3.23) are negligible if the spectral width of the pulse satisfies the condition  $\Delta\omega \ll \omega_0$ .

$$\beta(\omega) = \beta_0 + (\omega - \omega_0)\beta_1 + \frac{1}{2}(\omega - \omega_0)^2\beta_2 + \frac{1}{6}(\omega - \omega_0)^3\beta_3 + \dots, \quad (2.3.23)$$

- Their neglect is consistent with the **quasi-monochromatic assumption** used in the derivation of Eq. (2.3.22).
- If  $\beta_2 \approx 0$  for some specific values of  $\omega_0$ , it may be necessary to include the  $\beta_3$  term.
- Under the same conditions, we can use the approximation  $\Delta\beta \approx \Delta\beta_0$  in Eq. (2.3.25).

$$\Delta\beta(\omega) = \Delta\beta_0 + (\omega - \omega_0)\Delta\beta_1 + \frac{1}{2}(\omega - \omega_0)^2\Delta\beta_2 + \dots, \quad (2.3.25)$$

- After these simplifications in **Eq. (2.3.22)**, we take the **inverse Fourier transform** using

$$\frac{\partial \tilde{A}}{\partial z} = i[\beta(\omega) + \Delta\beta(\omega) - \beta_0]\tilde{A}, \quad (2.3.22)$$

$$A(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(z, \omega - \omega_0) \exp[-i(\omega - \omega_0)t] d\omega. \quad (2.3.26)$$

- During the **Fourier-transform operation**,  $\omega - \omega_0$  is replaced by the differential operator  $i(\partial/\partial t)$ .
- The resulting equation for  $A(z, t)$  becomes

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} = i\Delta\beta_0 A. \quad (2.3.27)$$

- The  $\Delta\beta_0$  term on the right side of **Eq. (2.3.27)** includes the effects of **fiber loss** and **nonlinearity**

- Using  $\beta(\omega) \approx n(\omega)\omega/c$  and assuming that  $F(x,y)$  in Eq. (2.3.20) does not vary much over the pulse bandwidth, Eq. (2.3.27) takes the form

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A = ir(\omega_0) |A|^2 A, \quad (2.3.28)$$

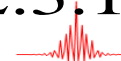
- Where the **nonlinear parameter**  $\gamma$  is defined as

$$\gamma(\omega_0) = \frac{n_2(\omega_0)\omega_0}{cA_{eff}}. \quad (2.3.29)$$

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$$\Delta\beta(\omega) = \frac{\omega^2 n(\omega)}{c^2 \beta(\omega)} \frac{\iint_{-\infty}^{\infty} \Delta n(\omega) |F(x,y)|^2 dx dy}{\iint_{-\infty}^{\infty} |F(x,y)|^2 dx dy}. \quad (2.3.20)$$

$$\Delta n = n_2 |E|^2 + \frac{i\tilde{\alpha}}{2k_0}. \quad (2.3.18)$$

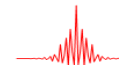




- In obtaining **Eq. (2.3.28)** the **pulse amplitude  $A$**  is assumed to be **normalized** such that  $|A|^2$  represents the **optical power**.
  - The quantity  $\gamma|A|^2$  (units of  $\text{m}^{-1}$ ) if  $n_2$  (units of  $\text{m}^2/\text{W}$ )
- The parameter  $A_{\text{eff}}$  (**effective mode area**) is defined as

$$A_{\text{eff}} = \frac{(\iint_{-\infty}^{\infty} |F(x, y)|^2 dx dy)^2}{\iint_{-\infty}^{\infty} |F(x, y)|^4 dx dy}. \quad (2.3.30)$$

- Its **evaluation** requires the use of **modal distribution  $F(x, y)$**  for the fundamental fiber mode.
- $A_{\text{eff}}$  depends on **fiber parameters** such as the **core radius** and the **core–cladding index difference**.
- If  $F(x, y)$  is approximated by a **Gaussian distribution**,  $A_{\text{eff}} = \pi w^2$ .



- The **width parameter**  $w$  depends on the  $V$  parameter of fiber and can be obtained from **Figure 2.1** or **Eq. (2.2.10)**.
  - $A_{\text{eff}}$  can vary in the range of **1 to 100  $\mu\text{m}^2$**  in the **1.5- $\mu\text{m}$**  region (depending on the fiber design)
  - $\gamma$  takes values in the range **1–100  $\text{W}^{-1}/\text{km}$**  (if  $n_2 \approx 2.6 \times 10^{-20} \text{ m}^2/\text{W}$  ).
- For **highly nonlinear fibers**,  $A_{\text{eff}}$  is reduced intentionally to enhance the nonlinear effects.

$$V = p_c a = k_0 a (n_1^2 - n_c^2)^{1/2}, \quad (2.2.10)$$

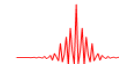
➤ **Eq. (2.3.28)** describes propagation of **picosecond optical pulse** in single-mode fibers

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A = ir(\omega_0) |A|^2 A, \quad (2.3.28)$$

- It is related to the **nonlinear Schrödinger (NLS) equation** and it can be reduced to that form **under certain conditions**.
- It includes the effects
  - of **fiber losses** through  $\alpha$ ,
  - of **chromatic dispersion** through  $\beta_1$  and  $\beta_2$ ,
  - of **fiber nonlinearity** through  $\gamma$ .

- Briefly, the **pulse envelope** moves at the **group velocity**  $v_g \equiv 1/\beta_1$ , while the effects of **group-velocity dispersion (GVD)** are governed by  $\beta_2$ .
- The **GVD** parameter  $\beta_2$  can be **positive** or **negative** depending on whether the wavelength  $\lambda$  is below or above the **zero dispersion wavelength**  $\lambda_D$  of the fiber
- In the **anomalous-dispersion regime** ( $\lambda > \lambda_D$ ),  $\beta_2$  is negative, and the fiber can support **optical solitons**.
- In **standard silica fibers** (the change in sign occurring in the vicinity of  $1.3 \mu\text{m}$ )
  - $\beta_2 \sim 50 \text{ ps}^2/\text{km}$  (visible region)
  - $\beta_2 \sim -20 \text{ ps}^2/\text{km}$  (near  $1.5 \mu\text{m}$ )
- The term on the right side of **Eq. (2.3.28)** governs the nonlinear effects of **self-phase modulation (SPM)**.

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A = i r(\omega_0) |A|^2 A, \quad (2.3.28)$$



# Nonlinear Shordinger Equation



➤ [https://www.youtube.com/watch?v=5vw0Csy\\_1rs](https://www.youtube.com/watch?v=5vw0Csy_1rs)

Mod-01 Lec-32 Introduction to Non-Linear Fiber Optics

$$\nabla \times \nabla \times \bar{E} = -\nabla \times \left\{ \mu_0 \frac{\partial \bar{H}}{\partial t} \right\} = -\mu_0 \frac{\partial}{\partial t} \{ \nabla \times \bar{H} \}$$

$$= -\mu_0 \frac{\partial}{\partial t} \left\{ \frac{\partial \bar{D}}{\partial t} \right\} = -\mu_0 \frac{\partial^2}{\partial t^2} \{ \epsilon_0 \bar{E} + \bar{P} \}$$

$$= -\mu_0 \frac{\partial^2 (\epsilon_0 \bar{E})}{\partial t^2} - \mu_0 \frac{\partial^2 \bar{P}}{\partial t^2}$$

$$\bar{P} = \underbrace{\epsilon_0 \chi^{(1)} \cdot \bar{E}}_{P_L} + \underbrace{\epsilon_0 \chi^{(3)} : \bar{E} \bar{E} \bar{E}}_{P_{NL}}$$

$$\nabla (\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = -\frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2} - \mu_0 \left\{ \frac{\partial^2 P_L}{\partial t^2} + \frac{\partial^2 P_{NL}}{\partial t^2} \right\}$$