Transverse Electric (TE) and Transverse Magnetic (TM) modes

- Since ν represents **angular dependence** of solution, the field solution to E_z when $v = 0$ will be **rotationally invariant**.
- The Equation (8) simplify to

$$\left[\frac{J_v'(\kappa a)}{\kappa J_v(\kappa a)} + \frac{K_v'(\gamma a)}{\gamma K_v(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_v'(\kappa a)}{\kappa J_v(\kappa a)} + \frac{k_0^2 n_{clad}^2 K_v'(\kappa a)}{\gamma K_v(\gamma a)}\right] = 0 \qquad \textbf{(13)}$$

$$\boxed{\frac{\beta^2 v^2}{a^2}\left[\frac{1}{\gamma^2} + \frac{1}{\kappa^2}\right]^2 = \left[\frac{J_v'(\kappa a)}{\kappa J_v(\kappa a)} + \frac{K_v'(\gamma a)}{\gamma K_v(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_v'(\kappa a)}{\kappa J_v(\kappa a)} + \frac{k_0^2 n_{clad}^2 K_v'(\kappa a)}{\gamma K_v(\gamma a)}\right]} \qquad \textbf{(8)}$$

- If the **first term** of Equation (13) is set to **zero**, then A must also be zero to keep the magnitude of B in Equation (11) finite

$$B = \frac{j\nu\beta}{\omega\mu a}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]^{-1} A \qquad \textbf{(11)}$$

- If A=0, then E_z=0, and the electric field will be transverse. Such modes are called **TE modes**

$$E_z(r,\phi,z) = AJ_\nu(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c.$$

$$H_z(r,\phi,z) = BJ_\nu(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c..$$

$$\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_0^2 n_{clad}^2 K_\nu'(\kappa a)}{\gamma K_\nu(\gamma a)}\right] = 0 \qquad \textbf{(13)}$$

- If the **second term** in Equation (10) is zero, then the amplitude B will be zero, and the **longitudinal component** of the *H* field will be zero.

$$B = \frac{j\omega a}{\beta v}\left[\frac{n_{core}^2 J_v^{'}(\kappa a)}{\kappa J_v(\kappa a)} + \frac{n_{clad}^2 K_v^{'}(\kappa a)}{\gamma K_v(\gamma a)}\right]\left[\frac{1}{\gamma^2} + \frac{1}{\kappa^2}\right]^{-1} A \qquad \textbf{(12)}$$

- Such modes are called **TM modes**
- If ν=0, the allowed modes will be either TE or TM

$$E_z(r,\phi,z) = AJ_v(\kappa r)e^{jv\phi}e^{-j\beta z} + c.c.$$

$$H_z(r,\phi,z) = BJ_v(\kappa r)e^{jv\phi}e^{-j\beta z} + c.c..$$

$$\left[\frac{J_v^{'}(\kappa a)}{\kappa J_v(\kappa a)} + \frac{K_v^{'}(\gamma a)}{\gamma K_v(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_v^{'}(\kappa a)}{\kappa J_v(\kappa a)} + \frac{k_0^2 n_{clad}^2 K_v^{'}(\kappa a)}{\gamma K_v(\gamma a)}\right] = 0 \qquad \textbf{(13)}$$

- These equation for the TE and TM modes can be simplified using the **Bessel function relations**

$$\frac{J_\nu'}{\kappa J_\nu} = \pm\frac{J_{\nu\mp1}}{\kappa J_\nu} \mp \frac{\nu}{\kappa^2}$$
$$\frac{K_\nu'}{\gamma K_\nu} = \mp\frac{K_{\nu\pm1}}{\gamma K_\nu} \mp \frac{\nu}{\gamma^2} \qquad \textbf{(14)}$$

- Consider first the **TE mode**. The first term of Equation (13) should be set equal to zero. Using the relation in Equation (14), the eigenvalue for TE modes become

$$\boxed{-\frac{J_1(\kappa a)}{\kappa J_0(\kappa a)} - \frac{K_1(\gamma a)}{\gamma K_0(\gamma a)} = 0} \qquad \textbf{(15)}$$

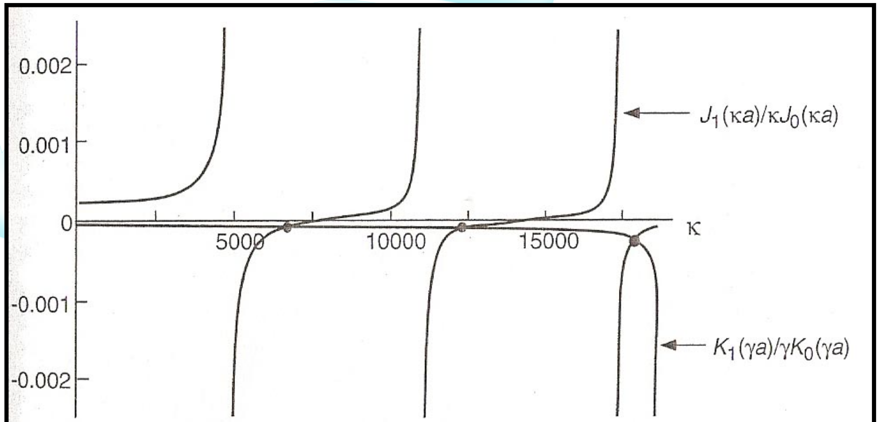


Figure 5.6 The eigenvalue equation is plotted against κ for a waveguide with core index 1.5, cladding index 1.45, and wavelength 1.3μm.

- As shown in the figure, it is clear that **no TE mode** will exist if ***κa*** **<2.405**.
- The **TE$_{01}$ mode** can only exist if ***κa*** **>2.405**, so the **cutoff condition** for theTE$_{01}$ mode is ***κa*** **=2.405**.
- The cutoff condition for the **TE$_{02}$ mode** occurs at the second root of the Bessel function, $J_0(\kappa a)$ which occurs **5.520**.

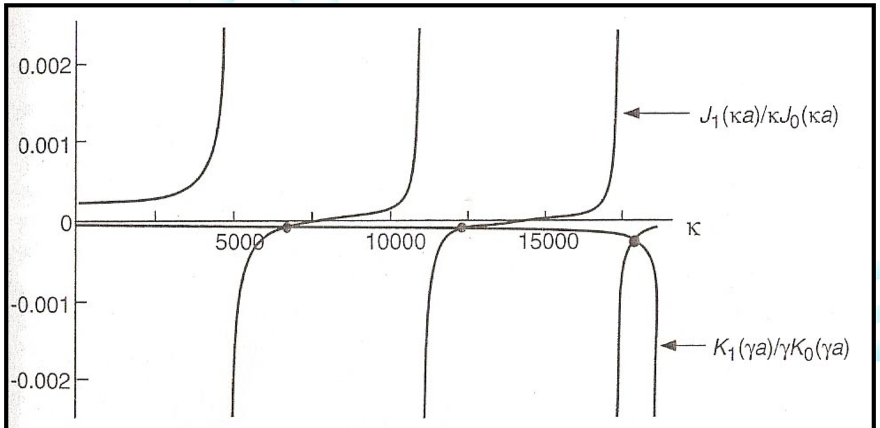
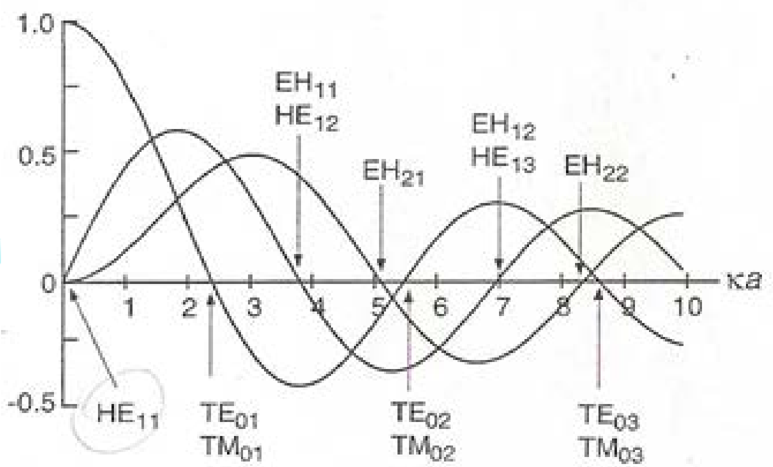


Figure 5.6 The eigenvalue equation is plotted against κ for a waveguide with core index 1.5, cladding index 1.45, and wavelength 1.3μm.



- For $\nu = 0$, these modes are **analogous to** the **transverse-electric (TE)** and **transverse-magnetic (TM) modes** of a **planar waveguide** because the axial component of the **electric field**, or the **magnetic field**, vanishes.

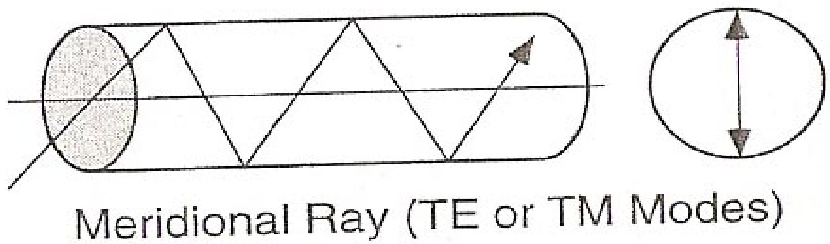


Figure 5.7 A meridional ray zigzags down the fiber, passing through the origin. There is no angular rotation of the ray path as it propagates.

In the ray picture, these modes are called **skew rays**, they travel down the waveguide in a **screw-like pattern**

Hybrid modes

- For **ν□0**, the values of β will correspond to modes which have finite components of both E_z and H_z and are therefore neither TE or TM modes
- The modes called EH or HE modes, depending on the relative magnitude of the longitudinal E and H components

For r <a

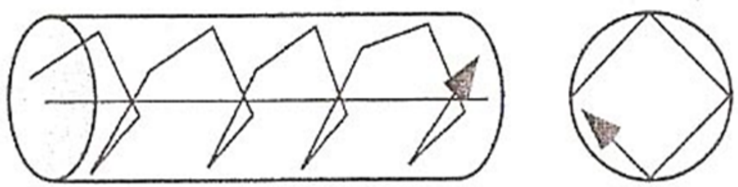
core

$$E_z(r,\phi,z) = AJ_\nu(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c.$$

$$H_z(r,\phi,z) = BJ_\nu(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c..$$

- ✓ If **A=0** then the mode is called a **TE mode**
- ✓ If **B=0** then the mode is called a **TM mode**
- ✓ If **A>B** then the mode is called an **HE mode** (E_z dominares H_z)
- ✓ If **A<B** then the mode is called an **EH mode** (H_z dominares E_z)

- Both **EH** and **HE modes** are called **hybrid modes** because they have both **longitudinal** *H* and *E* components in the waveguide.
- The EH and HE modes exist only for ν□1, so they **have azimuthal structure**.
- In the ray picture, these modes are called **skew rays**, they travel down the waveguide in a **screw-like pattern**
- The EH and HE mode have complicated field. The patterns are not only difficult to determine, but they are hard to visualize.



Skew Ray (EH or HE Mode)

Figure 5.8 A skew ray travels in a spiral path down the fiber. The ray does not go through the origin.

Linear Polarized modes (LP mode)

- For many practical optical fibers, the **core** and **cladding index** are nearly identical. ($n_{core} \approx n_{clad} = n$)
- In the **weakly guiding approximation** ($n_{core} \approx n_{clad} = n$) , Equation (8) reduce to Equation (15)

$$\frac{\beta^2 v^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J_v'(\kappa a)}{\kappa J_v(\kappa a)} + \frac{K_v'(\gamma a)}{\gamma K_v(\gamma a)}\right]^2 k_0^2 n^2 \qquad \textbf{(15)}$$

- Noting that if $n_{core} = n_{clad}$, then $\beta^2 = k_0^2 n^2$, and these terms can be canceled from both sides.
- Taking advantage of some **Bessel function identities**, simplifies Eq. (15), leaving only

$$\frac{J_{v\mp 1}(\kappa a)}{\kappa J_v(\kappa a)} = \mp \frac{K_{v\pm 1}(\gamma a)}{\gamma K_v(\gamma a)}$$

- These are the characteristic equations for the EH (top sign) and HE modes (bottom sign)

- A little more manipulation with Bessel function reduce these two equations into one single equation

$$\boxed{\kappa \frac{J_{j-1}(\kappa a)}{J_j(\kappa a)} = -\gamma \frac{K_{j-1}(\gamma a)}{K_j(\gamma a)}}$$

- The indices define the mode as follows：

***j* = 1**	**TE, TM modes**
***j* = *v*+1**	**EH$_v$ modes**
***j* = *v*-1**	**HE$_v$ modes**

- More than one mode has the **same eigenvalue**, different mode are **degenerate**
- In the **weakly guiding approximation**, the **TE$_{0m}$** is degenerate with the **TM$_{0m}$** modes
- They will have the same eigenvalue, β, and will propagate at the same velocity.
- The **HE$_{\nu+1,m}$** modes and **EH$_{\nu-1,m}$** modes are degenerate.

- Since **degenerate mode** travel at the **same velocity**, this degeneracy in β makes it possible to define stable superposition of different mode.
- Certain combinations of **degenerate modes** can be found which are **linearly polarized**
- The superpositions are primarily transverse, meaning E_z is negligible.
- The **designation** and **construction** of an LP mode is as follow:

LP_{1m}	→	**Sum of TE_{0m}, TM_{0m}, and HE_{2m} modes**
$LP_{\nu m}$	→	**Sum of $HE_{\nu+1,m}$ and $EH_{\nu-1,m}$ modes**
LP_{0m}	→	**HE_{1m} mode only (special case)**

Practical advantage of the LP modes

- First, the LP mode provide an easy way to visualize the structure of the **guided modes**.
  - Most of the energy is stored in the **transverse field**, there are no complications from energy stored in the longitudinal terms

- Second, the **LP modes** represent **actual energy distributions** that a **polarized source** would excite in a fiber.

- Finally, LP modes allow for a **simplified characteristic equation** that can be solved with **straightforward** or **graphical techniques**

Disadvantages of LP modes

- LP mode are **not true mode**, but are a superposition of slightly nondegenrate mode
  - The individual EH, HE, TM, and TE modes travel at ***slightly* different velocities**
  - The polarization of **initial superposition** will change as the mode propagate down the axis of guide
- The **LP mode** are only an **approximation of the true mode structure** of the guide
- They allow a simple way to visualize the mode, and describe the **actual field patterns** excited by the real sources

- The **wave equation** for $\tilde{E}_z$ is easily solved by using the method of **separation of variables**, resulting in the following **general solution**:

$$\boxed{\tilde{E}_z(r,\omega) = A(\omega)F(\rho)\exp(im\phi)\exp(i\beta z), \qquad (2.2.3)}$$

A: depends only on the frequency ω,

β: the propagation constant,

m : an integer,

- ***F*(*ρ*)** is the solution of **Bessel functions.**

$$\boxed{\frac{d^2F}{d\rho^2} + \frac{1}{\rho}\frac{dF}{d\rho} + \left(n^2k_0^2 - \beta^2 - \frac{m^2}{\rho^2}\right)F = 0, \qquad (2.2.4)}$$

$\boldsymbol{n = n_1}$ ($\rho \le a$,fiber of **core radius** a)

$\boldsymbol{n = n_c}$ ($\rho > a$)

$$\frac{\partial^2 \tilde{\mathrm{E}}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \tilde{\mathrm{E}}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \tilde{\mathrm{E}}}{\partial \phi^2} + \frac{\partial^2 \tilde{\mathrm{E}}}{\partial z^2} + n^2k_0^2\tilde{\mathrm{E}} = 0, \qquad (2.2.1)$$

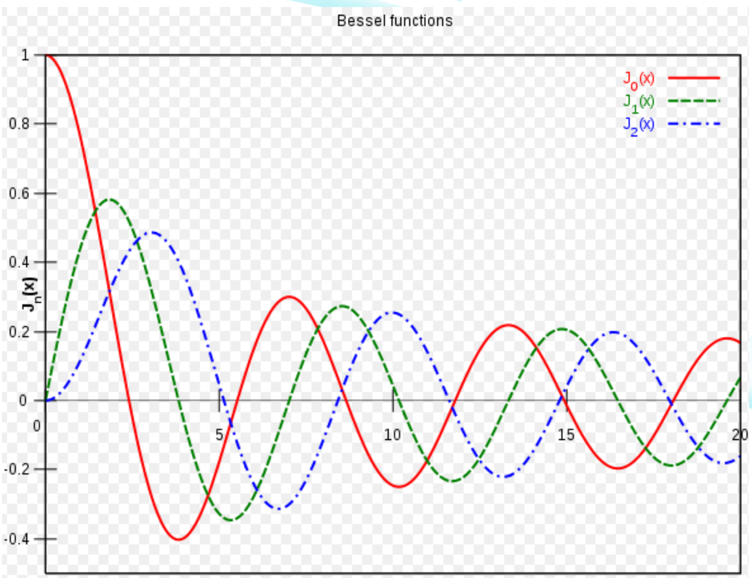
- Its **general solution** inside the core can be written as

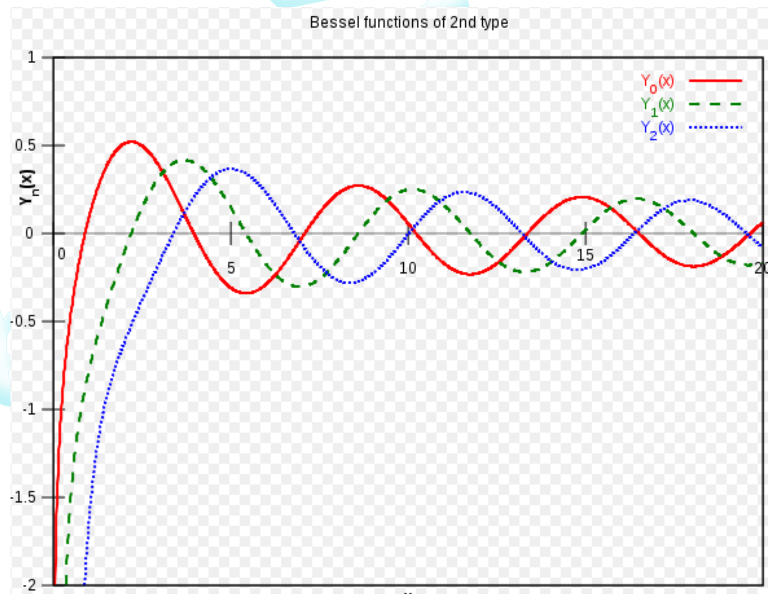
$$F(\rho) = C_1 J_m(p\rho) + C_2 N_m(p\rho), \qquad (2.2.5)$$

$J_m(x)$: Bessel function

$N_m(x)$: Neumann function

$$\boldsymbol{p = (n_1^2 k_0^2 - \beta^2)^{1/2}}$$





- The constants C_1 and C_2 are determined using the **boundary conditions**.
- $C_2 = 0$ for a **physically meaningful solution** ($N_m(p\rho)$ has a **singularity** at $\rho = 0$)
- The constant C_1 can be absorbed in A appearing in Eq. (2.2.3). Thus,

$$F(\rho) = J_m(p\rho), \qquad \rho \leq a. \tag{2.2.6}$$

$$p = (n_1^2 k_0^2 - \beta^2)^{1/2}$$

- In the **cladding region** ($\rho \geq a$), the **modified Bessel function** K_m represents the solution $F(\rho)$ such that it **decays exponentially** for large ρ.

$$F(\rho) = K_m(q\rho), \qquad \rho \geq a, \tag{2.2.7}$$

$$q = (\beta^2 - n_c^2 k_0^2)^{1/2}$$

- The same procedure can be followed to obtain the **magnetic field** component H_z.

- The **boundary condition** that the **tangential components** of $\tilde{E}$ and $\tilde{H}$ be continuous across the **core-cladding interface** requires that $\tilde{E}_z$, $\tilde{H}_z$, $\tilde{E}_\varphi$, and $\tilde{H}_\varphi$ be the **same** when $\boldsymbol{\rho = a}$ is approached from inside or outside the core.
- The **equality** of these field components at $\boldsymbol{\rho = a}$ leads to an eigenvalue equation whose solutions determine the **propagation constant** β for the fiber modes.
- The eigenvalue equation:

$$\boxed{\left[\frac{J_m'(pa)}{pJ_m(pa)} + \frac{K_m'(qa)}{qK_m(qa)}\right]\left[\frac{J_m'(pa)}{pJ_m(pa)} + \frac{n_c^2}{n_1^2}\frac{K_m'(qa)}{qK_m(qa)}\right] = \left(\frac{m\beta k_0(n_1^2-n_c^2)}{an_1p^2q^2}\right)^2, \quad (2.2.8)}$$

a prime denotes differentiation with respect to the argument

- we used the **important relation**

$$\boxed{p^2+q^2=(n_1^2-n_c^2)k_0^2. \qquad (2.2.9)}$$

- The **eigenvalue equation** (2.2.8) in general has **several solutions** for β for each **integer value** of m.

$$\left[\frac{J'_m(pa)}{pJ_m(pa)}+\frac{K'_m(qa)}{qK_m(qa)}\right]\left[\frac{J'_m(pa)}{pJ_m(pa)}+\frac{n_c^2}{n_1^2}\frac{K'_m(qa)}{qK_m(qa)}\right]=\left(\frac{m\beta k_0(n_1^2-n_c^2)}{an_1p^2q^2}\right)^2, \quad (2.2.8)$$

- It is **customary** to express these solutions by β_{mn}, where **both *m* (angular mode number)** and ***n* (radial mode number)** take **integer values**.
- Each **eigenvalue** β_{mn} corresponds to one **specific mode** supported by the fiber.
- The corresponding **modal field distribution** is obtained from Eq.(2.2.3).

$$\tilde{E}_z(r,\omega)=A(\omega)F(\rho)\exp(im\phi)\exp(i\beta z), \qquad (2.2.3)$$

- There are **two types** of fiber modes, designated as **HE$_{mn}$** and **EH$_{mn}$**.

- For ***m* = 0**, these modes are analogous to the **transverse-electric (TE)** and **transverse-magnetic (TM) modes** of a planar waveguide because the **axial component** of the **electric field**, or the **magnetic field**, vanishes.
- However, for ***m* > 0**, fiber modes become hybrid, i.e., all six components of the electromagnetic field are nonzero.

2.2.2 Single Mode Condition

- The **number of modes** supported by a **specific fiber** at a **given wavelength** depends on its design parameters,
  - the **core radius** *a*
  - the **core-cladding index difference** $n_1 - n_c$.
- An important parameter for each **mode** is its **cut-off frequency**. This frequency is determined by the condition $q = 0$.
- The value of $\boldsymbol{p}$ when $\boldsymbol{q = 0}$ for a given mode determines the **cut-off frequency** from Eq. (2.2.9).

$$p^2 + q^2 = (n_1^2 - n_c^2)k_0^2. \qquad (2.2.9)$$

- It is useful to define a a **normalized frequency** *V*

.

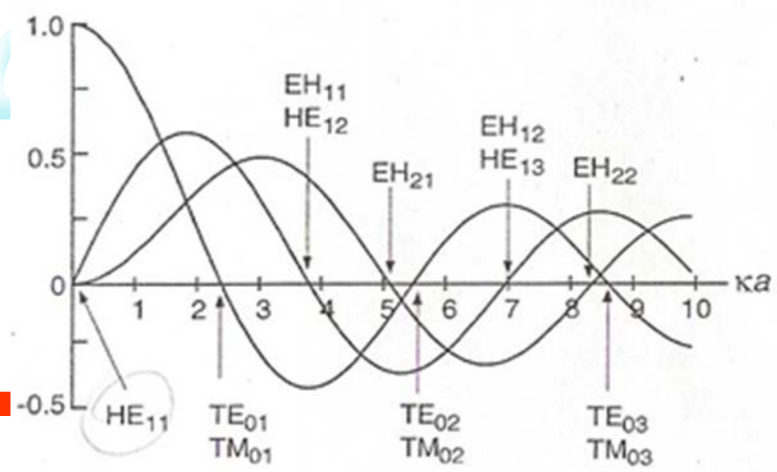
$$\boxed{V = p_c a = k_0 a (n_1^2 - n_c^2)^{1/2}, \qquad (2.2.10)}$$

p_c is obtained from Eq. (2.2.9) by setting $\boldsymbol{q = 0}$

- The **eigenvalue equation** (2.2.8) can be used to determine the values of V at which different modes reach **cut-off**.

$$\left[\frac{J_m'(pa)}{pJ_m(pa)}+\frac{K_m'(qa)}{qK_m(qa)}\right]\left[\frac{J_m'(pa)}{pJ_m(pa)}+\frac{n_c^2}{n_1^2}\frac{K_m'(qa)}{qK_m(qa)}\right]=\left(\frac{m\beta k_0(n_1^2-n_c^2)}{an_1p^2q^2}\right)^2, \quad (2.2.8)$$

- Since we are interested mainly in **single-mode fibers**, we limit the discussion to the **cut-off condition** that allows the fiber to support only **one mode (single-mode fibers,)**.
- A **single-mode fiber** supports only the **HE_{11} mode (fundamental mode)**.
- All other modes are beyond **cut-off** if the parameter $V < V_c$, where V_c is the **smallest solution** of $J_0(V_c) = 0$ or $V_c \approx 2.405$.



- Typically, **microbending losses increase** as V/V_c **becomes small**.
- In practice, fibers are designed such that V is **close** to V_c.

- The **cut-off wavelength** λ_c for **single-mode fibers** can be obtained by using $k_0 = 2\pi/\lambda_c$ and **V = 2.405** in Eq.(2.2.10).

$$\boxed{V = p_c a = k_0 a (n_1^2 - n_c^2)^{1/2}, \qquad (2.2.10)}$$

- For a **typical value** of index difference $n_1 - n_c = 0.005$, $\lambda_c = 1.2\ \mu$m and $a = 4\ \mu$m,
  - such a fiber supports a single mode only for $\lambda > 1.2\ \mu m$.
- In the **visible region**, **core radius** should be below **2 μm** for a fiber to support a **single mode**.

2.2.3 Characteristics of the Fundamental Mode

- The **field distribution *E*(r, t)** corresponding to the **$\mathbf{HE_{11}}$** mode has **three Nonzero components (either $\mathbf{E_x}$ or $\mathbf{E_y}$ dominates)**
  - Cylindrical coordinates E_ρ , E_φ , and E_z
  - Cartesian coordinates E_x , E_y , and E_z

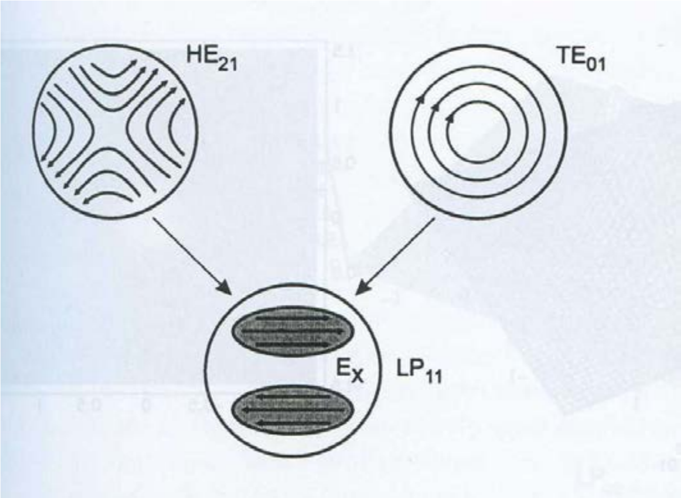
- To a good **degree of approximation**, the **fundamental fiber mode** is **linearly polarized** in either the ***x*** or ***y*** direction depending on whether E_x or E_y dominates

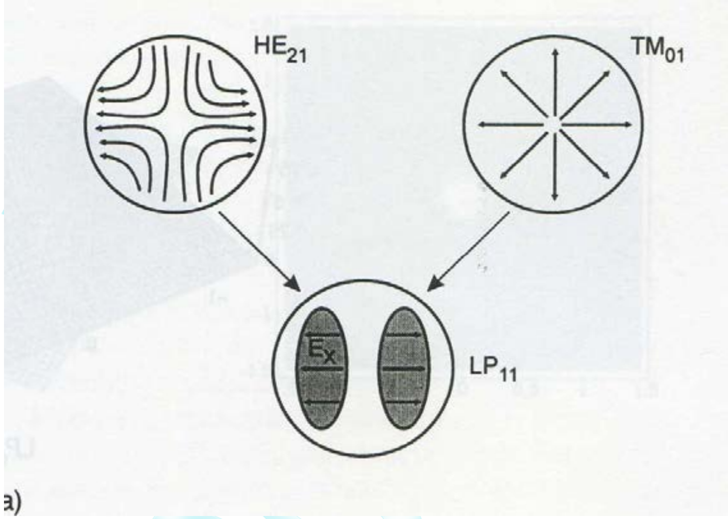
- Even a **single-mode fiber** is **not truly single mode** because it can support **two modes of orthogonal polarizations.**

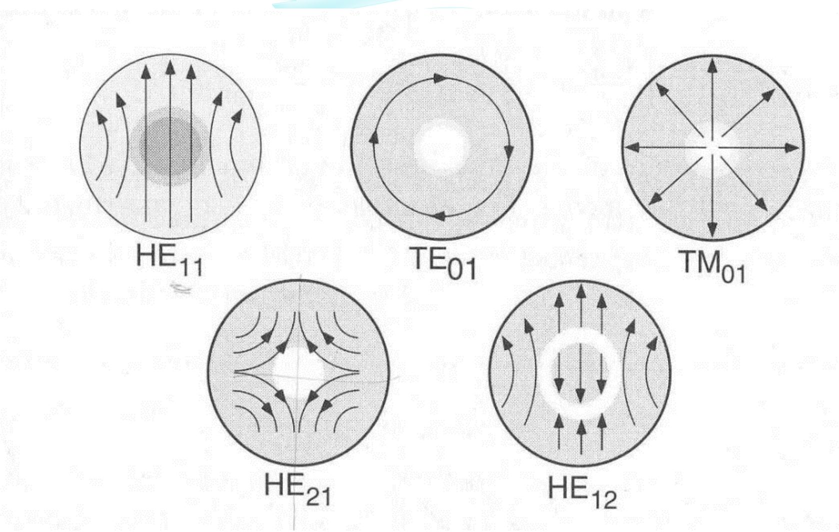
- The **fundamental mode HE_{11}** corresponds to **LP_{01}** (**LP_{mn}** denote the **linearly polarized modes**)
- Under **ideal conditions**, the **two orthogonally polarized modes** of a single-mode fiber are **degenerate** (with the **same propagation constant**).
- Certain combinations of **degenerate modes** can be found => **linearly polarized**

LP_{1n}	→	**Sum of TE_{0n}, TM_{0n}, and HE_{2n} modes**
LP_{mn}	→	**Sum of $HE_{m+1,n}$ and $EH_{m-1,n}$ modes**
LP_{0n}	→	**HE_{1n} mode only (special case)**

- The LP_{11} mode is a superposition of the $\mathbf{TE_{01}}$ and $\mathbf{HE_{21}}$ modes.
- Note that the LP mode is **linearly polarized**, in contrast to the electric fields of the **two constituent modes**.







Five different modes in a step-index circular waveguide

- **Irregularities** such as **random variations** in the **core shape** and **size** along the **fiber length break this degeneracy** slightly
  - **Mix** the **two polarization components** randomly, and **scramble** the **polarization** of the **incident light** as it propagates down the fiber.
- **Polarization-preserving fibers** can maintain the **linear polarization** if the light is launched with its polarization **along one of the principal axes** of the fiber.

- Assuming that the **incident light** is **polarized** along a **principal axis** (chosen to coincide with the **x axis**), the **electric field** for the fundamental fiber mode **HE$_{11}$** is approximately given by

$$\tilde{\mathbf{E}}(\mathbf{r},\omega)=\hat{x}\{A(\omega)F(x,y)\exp[i\beta(\omega)z]\} \qquad \textbf{(2.2.11)}$$

$A(\omega)$: Normalization constant; $\beta(\omega)$: Propagation constant

- $F(x, y)$: **transverse distribution**
  - Inside the core

$$F(x,y)=J_0(p\rho),\quad \rho\le a \qquad \textbf{(2.2.12)}$$

  - Outside the core

$$F(x,y)=(a/\rho)^{1/2}J_0(pa)\exp[-q(\rho-a)],\quad \rho\ge a \qquad \textbf{(2.2.13)}$$

$$\rho=(x^2+y^2)^{1/2} \quad \text{(Radial distance)}$$

- $K_m(q\rho)$ in Eq. (2.2.7) was approximated by the leading term in its **asymptotic expansion** and a **constant factor** was added to ensure the equality of $F(x,y)$ at $\rho = a$.

- Assuming that the **incident light** is **polarized** along a **principal axis** (chosen to coincide with the **x axis**), the **electric field** for the fundamental fiber mode $\mathbf{HE_{11}}$ is approximately given by

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \hat{x}\{A(\omega)F(x,y)\exp[i\beta(\omega)z]\} \qquad \textbf{(2.2.11)}$$

$A(\omega)$: Normalization constant; $\beta(\omega)$: Propagation constant

- The propagation constant β(ω) in Eq. (2.2.11) is obtained by solving the **eigenvalue equation** (2.2.8).

- The **frequency dependence of** $\beta(\omega)$ results
  - Material dispersion: frequency dependence of n_1 and n_c
  - Waveguide dispersion: frequency dependence of p

- **Material dispersion** generally **dominates** unless the **light wavelength** is close to the **zero-dispersion wavelength**.

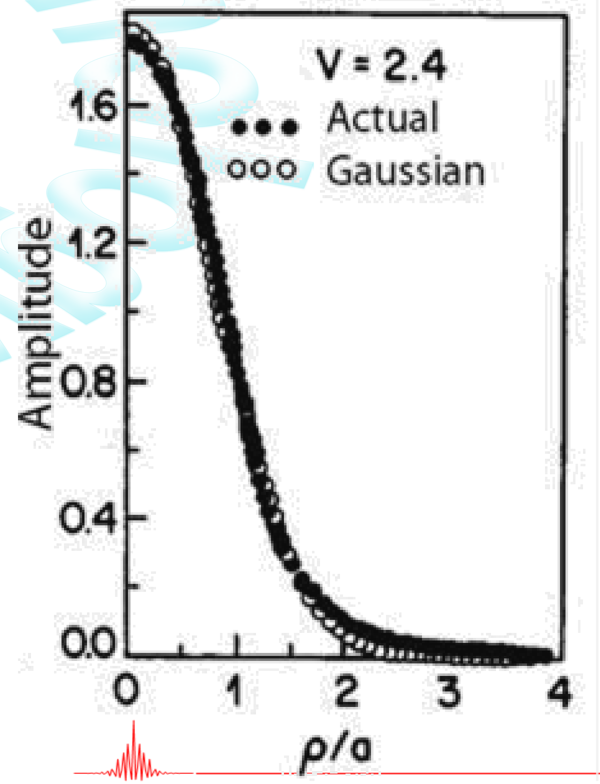
- The effective mode index is related to β by $n_{eff} = \beta/k_0$

- As the use of **modal distribution $F(x,y)$** in given by Eqs. (2.2.12) and (2.2.13) is cumbersome in practice
- The **fundamental fiber mode** is often approximated by a **Gaussian distribution** of the form

$$F(x, y) \approx \exp[-(x^2+y^2)/w^2] \qquad (2.2.14)$$

  - **w：width parameter**, determined by **curve fitting** or by following a **variational procedure**.

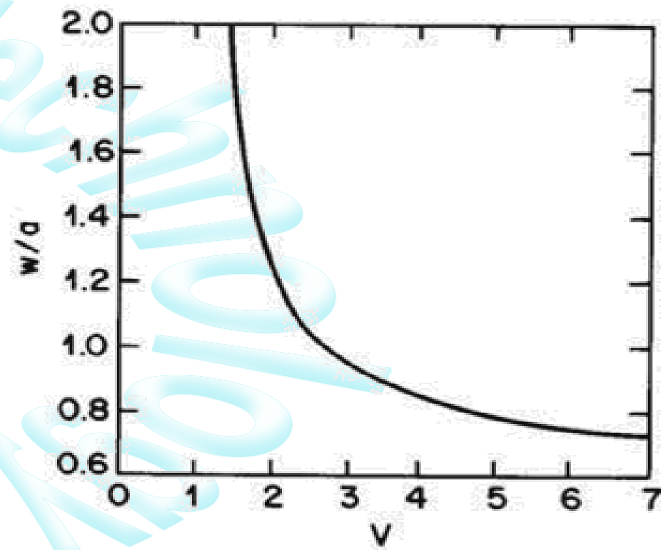
- For a **specific value V=2.4,** the comparison of the **actual field** distribution with the **fitted Gaussian**
- The **quality** of fit is generally quite good, particularly for **V** values in the neighborhood of **2**.



- Figure shows that $w \approx a$ for **V=2**, indicating that the **core radius** provides a **good estimate** of *w* for **telecommunication fibers** for which **V≈2.**
- For $V < 1.8$, *w* can be significantly larger than *a*
- When **1.2 <V < 2.4** (accurate to within **1%**)

$$w/a \approx 0.65 + 1.619V^{-3/2} + 2.879V^{-6}$$

(2.2.15)



- This expression is of considerable practical value as it expresses the mode width in terms of a **single fiber parameter V**.